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THE PROPAGATION OF ERRORS, FLUCTUATIONS AND TOLERANCES

BASIC GENERALIZED FORMULAS

by

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THE PROPAGATION OF ERRORS, FLUCTUATIONS AND TOLERANCES

BASIC GENERALIZED FORMULAS

ABSTRACT

The classical formulas for the "propagation of error" permit the approximate calculation of the variance of a function of variables whose variability is known. The adequacy of its approximation has often been doubted. Generalized formulas are here obtained, not only for average values and variances, but also for the next two cumulants.

The classical formula turns out to be better than expected -- and further approximations can be computed when necessary.

Expressing the individual variables in well chosen terms simplifies the generalized formulas and increases the accuracy of the classical one.

Higher cocumulants are introduced to aid the algebraic work of derivation.

The classical formula for the "propagation of error", where $y = f(x_1, x_2, \ldots, x_k)$, and the x_a suffer independent errors or fluctuations, is

$$\operatorname{var} y \approx \Sigma \left(\frac{\partial f}{\partial x_a}\right)_o^2 \operatorname{var} x_a = \Sigma f_a^2 \operatorname{var} x_a$$

where the subscript o means that $x_1, x_2, \ldots x_k$ are at their average values and where we have written f_a for the value of the a-th partial derivative at this average point. If the errors or fluctuations are correlated, this generalizes to

$$\operatorname{var} y \approx \Sigma\left(\frac{\partial f}{\partial x_a}\right)_o^2 \operatorname{var} x_a + 2\ \Sigma^*\left(\frac{\partial f}{\partial x_a}\right)_o \left(\frac{\partial f}{\partial x_b}\right)_o \operatorname{cov} \left\{x_a, x_b\right\}$$

where Σ^* indicates summation over all distinct terms with apparently different subscripts actually different. (See Section 38 for further details). These formulas are exact when y is a linear function of the x's.

$$y = a_o + a_1 x_1 + a_2 x_2 + \ldots + a_k x_k$$

and inexact otherwise. They have frequently been used for predicting the variation in a response y due to errors or to within-tolerance fluctuations in the x's. There has been a tendency to regard their use with suspicion because of this inexactness. There has been no indication of the extent of inaccuracy to hold this suspicion in proper check.

The present memorandum develops formulas, formally correct through terms of order σ^5, for the average value of y, its variance, and its next two higher cumulants. The most important conclusion is that the classical propagation formula is much better than seems to be usually realized. Examples indicate that it is quite likely to suffice for most work. The generalized formulas allow us (i) to check the accuracy of the classical formula, and (ii) to obtain much more accurate results in the few cases where they may be needed.

While the classical formula is often quite accurate when used in the terms in which the problem originally appears, it is often possible to improve its accuracy by a better choice of terms. If each of the individual or component variables x_1, x_2, ..., x_k is expressed in terms of its partial effect on the response, or in such terms that its partial effect is linear, we shall say that the terms are well-chosen. Unless the design value of one or more x's has been chosen to make y a maximum or a minimum, it will usually be possible and practical to use well-chosen terms. In treating the delay time of a delay line made up of LC-sections, for example, we are led to express component variables in terms of the square-root of inductance or the square-root of capacitance. In treating the attenuation of a π-section attenuator, for further example, we are led to specify the shunt elements in terms of their conductances and the series element in terms of its resistance.

Keeping further terms will require attention to the skewness of distribution of the individual variables, measured conveniently and dimensionlessly by

$$\gamma_a = \frac{\text{ave } (x_a - \overline{x}_a)^3}{[\text{ave } (x_a - \overline{x}_a)^2]^{3/2}}$$

Using Σ^* again to indicate summation over all distinct terms with apparently different subscripts actually different, using γ_a and σ_a to describe the distribution of the a-th individual variable, and abbreviating the various partial derivatives of f evaluated when $x_1, x_2, \ldots, x_k$ are each at its average value as illustrated by

$$f_{aab} = \left(\frac{\partial^3 f}{\partial x_a \, \partial x_a \, \partial x_b}\right) \text{ all } x\text{'s at average values,}$$

we can write that propagation formula for the variance which applies when (i) fluctuations in the various x_a are independent, and (ii) the x_a have been expressed as to make partial effects linear quite simply as follows:

$$\begin{aligned}
\text{var } y = {} & \Sigma f_a^{\,2} \sigma_a^{\,2} \\
& + \Sigma^*(f_{aab} f_b + f_{ab}^{\,2} + f_a f_{abb}) \sigma_a^{\,2} \sigma_b^{\,2} \\
& + \Sigma^*(\tfrac{1}{2} f_a f_{aabb} + f_{ab} f_{aab} + \tfrac{1}{3} f_{aaab} f_b) \gamma_a \sigma_a^{\,3} \sigma_b^{\,2} \\
& + \text{terms of order} \geq \sigma^6
\end{aligned}$$

Various aspects of this formula deserve attention:

(1) There is no term of order σ^3. Under our assumptions the first correction term is two orders higher than the leading term. Thus the leading term does better than we might expect.

(2) The first correction term involves both second (f_{ab}) <u>and</u> third (f_{aab} and f_{abb}) derivatives on an equal footing and with equal coefficients. If we are to go beyond first derivatives, we should go to the third derivatives.

(3) Through terms in σ^5, which is usually further than we have any excuse to go, we need only the values of the σ_a and the γ_a. Fewer quantities enter than we might fear.

(4) Much, but not all, of this simplicity comes from expressing the individual variables in proper terms. The general formula would have 8 terms instead of 3. Correct choice of terms is very helpful but not essential.

The parallel formula for the average is

$$\text{ave } y = y_0$$

$$+ \Sigma^*(\tfrac{1}{4} f_{aabb})\sigma_a^2\sigma_b^2$$

$$+ \Sigma^*(\tfrac{1}{12} f_{aaabb})\gamma_a\sigma_a^3\sigma_b^2$$

$$+ \text{terms of order} \geq \sigma^6.$$

Analogous formulas for the third and fourth cumulants are given in the body of the memorandum. (Table 1, page II-6).

It is often possible to reduce or eliminate the first correction term in the variance formula by changing the terms in which the response is expressed. (In the examples cited this leads to multiples of (i) the cube of a constant less the delay time and (ii) the cube root of the attenuation plus a constant.) While the possible advantages deserve being kept in mind, the first correction term is usually small enough so that this refinement is unnecessary. (The general question of simplifications is discussed in Section 9, pages II. 9-11.)

If the individual variables are <u>not</u> necessarily expressed so as to make f_{aa}, f_{aaa}, etc. vanish, but <u>do</u> have independent fluctuations, the propagation formulas are less simple, but can be expressed as in Table A, following. (Similar formulas for propagation into higher cumulants are given on page I-8.) These formulas require more information about the fluctuations of the individual variables -- information provided dimensionlessly by the values of Γ_a and G_a, where

$$\Gamma_a = \frac{\text{ave } (x_a - \bar{x}_a)^4}{[\text{ave } (x_a - \bar{x}_a)^2]^2}$$

and

$$G_a = \frac{\text{ave}(x_a - \bar{x}_a)^5}{[\text{ave } (x_a - \bar{x}_a)^2]^{5/2}}$$

Table A

General Propagation Formulas for Individual Variables with Independent Fluctuations

$$\text{ave } y = y_o$$

$$+ \frac{1}{2} \Sigma f_{aa} \sigma_a^{2}$$

$$+ \frac{1}{6} \Sigma f_{aaa} \gamma_a \sigma_a^{3}$$

$$+ \frac{1}{24} \Sigma f_{aaaa} \Gamma_a \sigma_a^{4} + \frac{1}{4} \Sigma^* f_{aabb} \sigma_a^{2} \sigma_b^{2}$$

$$+ \frac{1}{120} \Sigma f_{aaaaa} G_a \sigma_a^{5} + \frac{1}{12} \Sigma^* f_{aaabb} \gamma_a \sigma_a^{3} \sigma_b^{2}$$

$$+ \text{terms of order} \geq \sigma^6$$

$$\text{var } y = \Sigma f_a^{2} \sigma_a^{2}$$

$$+ \Sigma f_a f_{aa} \gamma_a \sigma_a^{3}$$

$$+ \frac{1}{3} \Sigma f_a f_{aaa} \Gamma_a \sigma_a^{4} + \frac{1}{4} \Sigma h_{aa}^{2} (\Gamma_a - 1) \sigma_a^{4}$$

$$+ \Sigma^*(f_a f_{abb} + f_{ab}^{2} + f_{aab} f_b) \sigma_a^{2} \sigma_b^{2}$$

$$+ \frac{1}{12} \Sigma f_a f_{aaaa} G_a \sigma_a^{5} + \frac{1}{6} \Sigma f_{aa} f_{aaa} (G_a - \gamma_a) \sigma_a^{5}$$

$$+ \Sigma^*(\tfrac{1}{2} f_a f_{aabb} + \tfrac{1}{2} f_{aa} f_{abb} + f_{ab} f_{aab} + \tfrac{1}{3} f_{aaab} f_b) \gamma_a \sigma_a^{3} \sigma_b^{2}$$

$$+ \text{terms of order} \geq \sigma^6$$

(Note that the whole coefficient of $\sigma_a^2 \sigma_b^2$ is symmetrical so that only one of (a,b) and (b,a) is summed.)

Propagation formulas for the case where the fluctuations of the individual variables are not independent are of increased complexity, and will be reported later.

The computations leading to the formulas are simplified by the introduction for the next two higher cumulants of 'co-' quantities related to these cumulants as the covariance is related to the variance. They involve 3 and 4 arguments, respectively. (Definitions may be found in Section 19, pages V. 1-2).

The structure of the detailed account is indicated by the following table of contents. Results are stated and discussed in the first four parts, with details left to the remaining parts.

The examples have been treated both by "main strength and awkwardness", and by more polished methods, so as to give some idea both of what would be required for less simple examples and of how easily examples can be handled.

1. Introduction

Fluctuations in "individual" or "component" variables often combine to produce fluctuations in a "resultant", "system" or "overall" variable -- fluctuations in a response. Two classical examples are the combination ("propagation") of errors of individual operations in a physical measurement and the combination of within-tolerance fluctuations in a mechanical or electrical assembly. In these situations, as in a wide variety of others, it is often desired to relate the distributions of component fluctuations to the distribution of resultant fluctuations. For a long time, physics has used the formulas for "the propagation of error". With the rise of modern statistical theory, it has been natural to regard the propagation-of-error formulas as doubtful first approximations, and to try to avoid their use in situations of complex dependence, even at the cost of proposing great complications in study or experiment or even, perhaps, at the cost of seeking no answer to the problem.

Insufficient attention seems to have been given to the question of how accurate, or inaccurate, the propagation-of-error formulas would be, or to how they may be improved. The obvious first step is to obtain and study better approximations. When this is done, the classical formulas turn out to be better than most of us had suspected.

But more can be done. The typical problem is naturally expressed as follows:

Given the functional relation

$$z = h(w_1, w_2, \ldots, w_k) ,$$

and given information about the distributions of the individual w's which distributions have been so chosen as to make the values of z be distributed near z_o what is the probability that, given a "tolerance" + δ [or $-\varepsilon$ or both], that z will exceed $z_o + \delta$ [or fall below $z_o - \varepsilon$, or either]?

There is nothing in the expression of this problem which requires us to work in terms of z, rather than z^2, log z, $\sqrt{z}$ or z^3: We can translate the boundary value $z + \delta$ [or $z - \varepsilon$] into each of these scales, or any other. Similarly, we may replace w_1 by w_1^2, log w_1, or any other helpful function. And we may do the same with $w_2, w_3, \ldots, w_k$. Thus we may as well work with variables $y, u_1, u_2, \ldots, u_k$, a functional relation $y = f(u_1, u_2, \ldots, u_k)$, and a tolerance $y_o + \delta^*$ [or $y_o - \varepsilon^*$] if we gain by doing this. It will appear that we can increase the quality of approximation of the classical propagation-of-error formula in this way.

While the same techniques, concepts and insights could be applied to the combination of (statistically) dependent fluctuations, the problem of combining (statistically) <u>independent</u> fluctuations, -- which leads to simpler results -- is of sufficient importance to lead us to treat only that case. (After all, most practical problems with dependent fluctuations can be converted into problems with independent ones by pushing back to suitable variables -- often to variables earlier determined.)

I PROPAGATION

2. Moments

The basic tool of "propagation of error" has always been the "mean square deviation", or, as statisticians now say, the "variance". If we are to go to more detailed approximations, we must expect to supplement this with more complex quantities.

The analogy between probability distributions and mass distributions and the moments of force, area, etc. of mechanics, early led to the description of probability distributions in terms of "moments", either about zero

$$\mu_1' = \int z p(z)\, dz = \text{ave } z,$$

$$\mu_2' = \int z^2 p(z)\, dz = \text{ave } z^2,$$

$$\mu_3' = \int z^3 p(z)\, dz = \text{ave } z^3,$$

$$\mu_4' = \int z^4 p(z)\, dz = \text{ave } z^4$$

$$\cdots$$

or about the mean or average $\bar{z}$ or μ_1 (this notation is used as well as μ_1') of the given distribution

$$\mu_2 = \int (z - \bar{z})^2\, p(z)\, dz = \mu_2' - \mu_1{}^2 = \text{ave } (z - \bar{z})^2$$

$$\mu_3 = \int (z - \bar{z})^3\, p(z)\, dz = \mu_3' - 3\mu_1\mu_2' + 2\mu_1{}^3 = \text{ave } (z - \bar{z})^3$$

$$\mu_4 = \int (z - \bar{z})^4\, p(z)\, dz = \mu_4' - 4\mu_1\mu_3' + 6\mu_1{}^2\mu_2' - 3\mu_1{}^4 = \text{ave}(z-\bar{z})^4$$

$$\cdots$$

While the dimensions of μ_1 are the same as those of z, the dimensions of μ_2, μ_3, ... are the square, cube, ... of those dimensions. It is often convenient to make this clear by introducing another quantity of the dimensions of z, together with suitable dimensionless coefficients. This is most easily done, as has been done for a long time, by introducing the root-mean-square deviation, now commonly called the standard deviation, σ, where $\sigma^2 = \mu_2$, and expressing μ_3, μ_4, ... as suitable multiplies of σ^3, σ^4, ... Various systematic notations have been proposed for the dimensionless multipliers, but their use seems to lead to complexity of notation whenever several variables are involved. Consequently we shall here use the following rather unsystematic notation

$$\sigma^2 = \text{var } z = \text{ave } (z-\bar{z})^2 = \int (z-\bar{z})^2\, p(z)\, dz,$$

$$\gamma\sigma^3 = \text{ave } (z-\overline{z})^3 = \int (z-\overline{z})^3 \; p(z) \; dz,$$

$$\Gamma\sigma^4 = \text{ave } (z-\overline{z})^4 = \int (z-\overline{z})^4 \; p(z) \; dz,$$

$$G\sigma^5 = \text{ave } (z-\overline{z})^5 = \int (z-\overline{z})^5 \; p(z) \; dz.$$

We call γ, Γ, and G relative third, fourth and fifth moments.

3. Propagation into Averages and Variance.

If now we write σ_1, γ_1, Γ_1 and G_1 for the corresponding quantities associated with w_1, and, similarly σ_2, γ_2, Γ_2 and G_2 for those associated with w_2, and so on and on, we can write down the generalized propagation of error formulas for the case where

$$z = h(w_1, w_2, \ldots, w_k)$$

and the fluctuations in w_1, w_2, ..., w_k are independent. These formulas involve the partial derivatives of $h(w_1, w_2, \ldots, w_k)$, evaluated at the point where each w_a, $a = 1, 2, \ldots, k$ takes its average value. We denote such numerical values of partial derivatives -- such derivative values, as we shall say later -- by h_1, h_{23}, h_{111}, h_6, h_{aab}, ... where the number of subscripts shows the number of differentiations and the particular subscripts specify directly the subscripts of the w's with respect to which these differentiations were carried out.

If we write z_o for the value of z when all w_a are at their average values, then

$$\text{ave } z = z_o$$
$$+ \frac{1}{2} \Sigma h_{aa} \sigma_a^2$$
$$+ \frac{1}{6} \Sigma h_{aaa} \gamma_a \sigma_a^3$$
$$+ \frac{1}{24} \Sigma h_{aaaa} \Gamma_a \sigma_a^4 + \frac{1}{4} \Sigma^* h_{aabb} \sigma_a^2 \sigma_b^2$$
$$+ \frac{1}{120} \Sigma h_{aaaaa} G_a \sigma_a^5 + \frac{1}{12} \Sigma^* h_{aaabb} \gamma_a \sigma_a^3 \sigma_b^2$$
$$+ \text{terms of order} \geq \sigma^6$$

where the starred summation signs are to be interpreted as including each distinct term which does not identify subscripts once and only once. Thus $\Sigma^* \sigma_a^2 \sigma_b^2$ would include $\sigma_4^2 \sigma_5^2$ once (not as $\sigma_4^2 \sigma_5^2 + \sigma_5^2 \sigma_4^2$) but not σ_4^4, while $\Sigma^* \sigma_a^3 \sigma_b^2$ would include $\sigma_4^3 \sigma_5^2 + \sigma_5^3 \sigma_4^2$ (and not merely one of these) but not σ_4^5. This same convention on starred summation signs will be followed throughout. (Note that h_{aabb} is symmetric in a and b by its definition.)

Similarly, the variance is given by

$$\text{var } z = \Sigma h_a^2 \sigma_a^2$$
$$+ \Sigma h_a h_{aa} \gamma_a \sigma_a^3$$

(formula continues)

$$+ \frac{1}{3} \Sigma h_a h_{aaa} \Gamma_a \sigma_a^4 + \frac{1}{4} \Sigma h_{aa}^2 (\Gamma_a - 1) \sigma_a^4$$

$$+ \Sigma^* (h_a h_{abb} + h_{ab}^2 + h_{aab} h_b) \sigma_a^2 \sigma_b^\sigma$$

$$+ \frac{1}{12} \Sigma h_a h_{aaaa} G_a \sigma_a^5 + \frac{1}{6} \Sigma h_{aa} h_{aaa} (G_a - \gamma_a) \sigma_a^5$$

$$+ \Sigma^* (\frac{1}{2} h_a h_{aabb} + \frac{1}{2} h_{aa} h_{abb} + h_{ab} h_{aab}$$

$$+ \frac{1}{3} h_{aaab} h_b) \gamma_a \sigma_a^3 \sigma_b^2$$

$$+ \text{terms of order} \geq \sigma^6$$

The first term on the right hand side is, of course, the form which the classical propagation of error formulas take for the case of independent w's.

If we should be content to assume that the distribution of z were normal (= Gaussian), then we might answer our target question easily. For the tolerance δ corresponds to a standardized deviate of

$$\frac{(z_o + \delta) - \text{ave } z}{\sqrt{\text{var } z}}$$

which is, in case of normality, easily converted into the probability of exceeding δ by making one reference to a table of the cumulative normal distribution. However, such an assumption could sometimes prove most dangerous, since for example, $h_1(w_1, w_2, \ldots, w_k)$ and $h_2(w_1, w_2, \ldots, w_k)$, where $h_2 = 17(h_1 - z_o)^2$ cannot both be nearly normally distributed. We must be prepared to carry our formulas further, either so that we may show the normal approximation satisfactory, or so that we may do better, as circumstances dictate. It is natural to attempt this by using the higher moments of z.

4. Seminvariants or cumulants

There is much to gain by avoiding a frontal attack on these higher moments. First Thiele, and then R. A. Fisher, have vigorously pointed out the advantages, particularly when effects are being combined, of replacing the higher moments about the mean by another set of quantities, called seminvariants or cumulants, for which the notation κ_1, κ_2, κ_3, ... is usual. The first two of these are the "average" and the "variance". There has been no general agreement on similar names for the succeeding quantities, but the word "skewness" has long been associated with κ_3. Although, unfortunately, "skewness" has also been associated with various dimensionless quantities, we shall here call κ_3 itself the "skewness", with the natural 3-letter abbreviation ske z It is the last cumulant to reduce to a moment about the mean ($\kappa_3 \equiv \mu_3$). In qualitative terms it usually tells us about a dissymmetry of the distribution concerned -- positive skew corresponding to a "longer" tail -- one more slowly declining to zero -- on the right hand of the distribution than on the left -- negative skew corresponding to the mirror image of this situation.

The next, fourth, cumulant, $\kappa_4 = \mu_4 - 3\mu_2^2$, has long been connected with the word "kurtosis" -- a word which led even eminent statisticians to mnemonic devices. We shall use, as an interim measure, subject to a better suggestion, the term "elongation" and the abbreviation elo z. Positive elongation usually means that the tails (or possibly one dominant tail) of the distribution fall toward zero more slowly than is the case for (comparable) normal (Gaussian or Maxwellian) distributions. Thus distributions with tails decreasing like e^{-cz}, rather than like e^{-cz^2}, are almost sure to have positive elongation. If the tails are shorter than those of normal distributions, as notably in rectangular (sometimes called uniform) distributions, or in U-shaped distributions, the elongation is usually negative.

Various devices have been psoposed to make use of information from higher moments or cumulants in selecting better approximate distributions than the normal. Three broad classes deserve mention here, although none of them is available adequately packaged for easy use.

Methods based on transformation into terms of increased normality were strongly urged by Edgeworth in the early years of this century. However, no generally useful methods were developed. Transformation in the special situation with which we are now dealing is investigated in Parts IV, VIII and IX, with mildly discouraging results.

Methods based on series expansion in terms of successive derivatives of the normal distribution have a respectable antiquity, were modified by Edgeworth, and were brought into their most useful and easily usable form to date by Cornish and Fisher* in 1937.

Methods based on a family of curve shapes based on a certain differential equation with a number of adjustable parameters were introduced about the turn of the century by Karl Pearson. Practical use of these "Pearson curves" has tended to be onerous, until the appearance in 1951 and, in improved form, in 1954 of convenient tables of % points** in standard measure for distributions following Pearson curves. In instances where a standardized deviate falls somewhere near a tabulated value, those tables offer the handiest solution so for available, and their use, together with the use of Cornish-Fisher technique might well have been described in the present memorandum. In view of the fact, however, that a still more convenient packaging seems possible, all detailed discussion is being omitted.

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*E. A. Cornish and R. A. Fisher "Moments and cumulants in the specification of distributions" 4 Revue de Institut Intern. de Statistique 1-14 (1937) reprinted as paper 30 in R. A. Fisher, Contributions to Mathematical Statistics, New York, Wiley, 1950.

**Maxine Merrington and E. S. Pearson "Tables of the 5% and 0.5% points of Pearson curves (with argument β_1 and β_2) explored in Standard Measure" 38 Biometrika 4-10 (1951) also in improved form as Table 42 of Biometrika Tables for Statisticians, Volume 1, E. S. Pearson and H. O. Hartley editors, Cambridge University Press 1954.

Either the Cornish-Fisher or Merrington-Pearson technique starts conveniently from dimensionless ratios of cumulants. Those entering into the Cornish-Fisher formulas are

$$\gamma_1 = \frac{\text{ske } z}{[\text{var } z]^{3/2}} = \frac{\mu_3}{\mu_2^{3/2}}$$

and

$$\gamma_2 = \frac{\text{elo } z}{[\text{var } z]^2} = \frac{\mu_4}{\mu_2^2} - 3$$

both of which vanish for the normal distribution. Those entering into the Merrington-Pearson tables are

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{[\text{ske } z]^2}{[\text{var } z]^3} = \gamma_1^2,$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{\text{elo } z}{[\text{var } z]^2} = 3 + \gamma_2,$$

whose values for a normal distribution are 0 and 3. Both approaches make use of the standardized deviate, or deviate expressed in standard measure

$$\frac{(z_o + \delta) - (\text{ave } z)}{\sqrt{\text{var } z}}$$

relating this either to a modified deviate which may be referred to a unit normal distribution, or directly to the probability of exceeding $z + \delta$.

To keep the record clear, we must emphasize that both methods are only approximate, though in practice usually adequate. Experience with the Pearson curves near the % points so far tabulated has been excellent. In further view of the convenience of the Merrington-Pearson tables, their use for true probabilities of exceeding tolerance between, say, 0.3% and 8% is at present to be recommended. For probabilities greater than about 10% (and less than 90%) Cornish-Fisher treatment may be best. In either case, at the user's choice, the purpose of more detailed calculation may be either to indicate the adequacy of the normal approximation, or to provide a better, satisfactory, approximation.

5. Propagation into Skewness and Elongation

We can now state the analogous formulas for the leading terms in ske z, and elo z, namely

$$
\begin{aligned}
\text{ske } z = {} & \Sigma h_a^3 \gamma_a \sigma_a^3 \\
& + \tfrac{3}{2} \Sigma h_a^2 h_{aa} (\Gamma_a - 1) \sigma_a^4 + 6\Sigma^* h_a h_b h_{ab} \sigma_a^2 \sigma_b^2 \\
& + \tfrac{1}{2} \Sigma h_a^2 h_{aaa} (G_a - \gamma_a) \sigma_a^5 + \tfrac{3}{4} \Sigma h_a h_{aa}^2 (G_a - 2\gamma_a) \sigma_a^5 \\
& + 3 \Sigma^* h_a h_{ab}^2 (\gamma_a - 1) \sigma_a^3 \sigma_b^2 \\
& + \tfrac{3}{2} \Sigma^* (h_a^2 h_{abb} + 2h_{aa} h_{ab} h_b + 2h_a h_{aab} h_b) \gamma_a \sigma_a^3 \sigma_b^2 \\
& + \text{terms of order} \geq \sigma^6
\end{aligned}
$$

and

$$\text{elo } z = \Sigma h_a^{4}(\Gamma_a - 3)\sigma_a^{4}$$

$$+ 2\,\Sigma h_a^{3} h_{aa}(G_a - 4\,\gamma_a)\sigma_a^{5}$$

$$+ \text{terms of order} \geq \sigma^{6}$$

The detailed correctness of the results of this account, both above and in sections to come, owes much to careful checking and error finding by Miss M. S. Harold. Thanks go to her from the writer, both on his own behalf and on behalf of future users.

SIMPLIFICATION BY CHANGE OF TERMS FOR INDIVIDUAL VARIABLES

6. General

We come now to the uses of changes in the terms of study. The fraction of some population of electronic assemblies with a certain frequency greater than 400 cycles per second is, of course, exactly the same as the fractions (i) with the square root of the frequency greater than 20 $(\text{cycles/sec})^{1/2}$, or (ii) with the common logarithm of the frequency (in cycles per second) greater than 2.60206^{-}. If it is easier to work with one of these equal fractions rather than either of the others, it is clearly to our advantage to do so.

In some circumstances, where the use of unfamiliar terms of analysis is called "transformation", or even "transforming the data" there is, to the minds of some, an unfortunate flavor of "cooking". While that feeling is almost always (if not always!) unjustified, in the present situation there is no slightest excuse for a similar feeling. Suppose that individual assemblies are tested -- for attenuation, resistance, amplification, critical frequency, or what have you. And suppose further that the result is displayed by the motion of a needle across a scale. Certain points on that scale correspond to assemblies below the tolerance, others to assemblies above tolerance. And this need be changed in no way whatsoever if the numbers on the scale represent, for example:

(i) ohms

(ii) $\sqrt{\text{ohms}}$

(iii) logarithm of ohms

(iv) $(\text{ohms})^{-1/2}$

(v) $(\text{ohms})^{-1}$ = mhos

or any other similar scale. Just so long as the needle's position exactly at the tolerance is kept the same, just so long will we be dealing with the same problem.

All this has referred to changes in terms from z to some function y of z. The situation in a change of terms from some w_a to some function v_a of w_a is similar, but not identical. If we are to use either the classical, or the generalized, propagation formulas, we need to know something about the moments of the distribution of w_a in one instance, and those of the distribution of v_a on the other. This fact seems to have worried persons with a mathematical background and orientation far more than it should -- as is rather natural. If one starts to treat a situation by saying "Let us assume that we know the low moments of the w_a" and, presume that "we may treat the joint distribution of the w_a as multivariate normal", and sticks rigidly to these hypotheses, as would be appropriate in pure mathematics, then he must realize that he does not know the low moments of the v_a exactly, and, moreover, if the joint distribution of the w_a was exactly multivariate normal, then that of the v_a cannot be. If the first set of starting hypotheses was taken as gospel engraved on tablets of stone, then the transformation appears to weaken and distort the available knowledge. But in a practical situation this is only an appearance. One never knows the low moments exactly, and, although one may have fairly good estimates, one often does not. (In tolerance analysis situations the low moments are often of a "but what if" character, and are far from precise.) Moreover, if <u>approximate</u> normality could be assumed for the w's, it is not unlikely that <u>approximate</u> normality can be equally well assumed for the v's. We must never forget that, in applications of mathematics, exact hypotheses are usually only approximations to the real situations and that other approximations may be as good, or even better, as those we first made.

This apparently subversive, but actually usually unimportant, effect of transformation of individual variables is enhanced in dealing with propagation formulas. These formulas involve derivative values at "the average point", and the average value of v_a, where v_a is a given function of w_a, is not exactly the same as the given function of the average value of w_a. Thus the average point shifts under the transformation and the derivative values we need are not only for derivatives with respect to new variables, which we can evaluate by simple transformations, but are evaluated at a new point. If we had known the old average point exactly, then this loss in firmness would have been relevant, though probably not serious. But we usually do not know it exactly, and in practice we are not appreciably worse off.

Thus, while it is conceivable that we might be able to handle the moments of the w_a and evaluate derivatives at the w_a-average point -- and still not be able to do the same in terms of the v_a -- such a situation is most unlikely to occur. In practical situations, transformation of individual variables may usually be undertaken quite freely.

7. Transformation of the Individual (Component) Variables

We can discover these transformations by complex arguments if we wish, merely by letting w_a be some function of v_a, and finding successive conditions on the derivatives of this function to make more and more terms vanish. nd then finding a function with these derivatives. But we may also avoid all this.

Consider what happens if all w's except w_a are at their average values, then z depends on w_a alone, but usually not in a linear way. If we may reasonably define v_a so as to make this dependence linear, then in

$$z = g(v_1, v_2, \ldots, v_k)$$

the higher unmixed derivative values g_{aa}, g_{aaa}, g_{aaaa}, which we shall later refer to as successive derivative values, will all vanish, and the formulas will be simplified.

When may we reasonably do this? Anytime that z is a strictly monotonic (ever-increasing, or ever-decreasing) function of each w_a over its normal range of variation. For if we write $\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_k$ for the average values of $w_1, w_2, \ldots, w_k$ and

$$\psi_a(w_a) = h(\overline{w}_1, \ldots, \overline{w}_{a-1}, w_a, \overline{w}_{a+b}, \ldots, \overline{w}_k)$$

for the value of $z = g(w_1, w_2, \ldots, w_k)$ with all the variables, other than w_a, at their average values, then $\psi_a(w_a)$ is naturally described as, and called, the partial effect of w_a at the average point (really along the line through the average point for which $w_1, w_2, \ldots, w_{a-1}$,

$w_{a+1}, \ldots, w_k$ -- all w's except w_a -- are constant). If we could introduce $v_a^{(1)} = \psi_a(w_a)$ as a new variable, so that

$$z = g^{(1)}\left\{v_1^{(1)}, v_2^{(1)}, \ldots, v_k^{(1)}\right\}$$

describes the response, then the successive derivatives (after the first) of $g^{(1)}(-,-,\ldots,-)$ with respect to $v_a^{(1)}$, evaluated at the v-values corresponding to the w-average point will vanish. This will be so because $z = v_a^{(1)}$ so long as the $v^{(1)}$'s other than $v_a^{(1)}$ are at the values corresponding to this point, since the w's other than w_a will all be at their average value. Two difficulties then remain: First, the transformation may not be permissible. Second, we wish the successive derivatives to vanish at the v-average point rather than at the w-average point.

The transformation will fail to be permissible if two values of w_a correspond to the same value of $\psi_a(w_a)$. In practice we require such single-valuedness of the inverse only for values of w_a reasonably near $\overline{w}_a$. Such a failure of uniqueness will ordinarily only occur when, because of design (or perhaps by accident) the nominal value of w_a has been chosen to make z exactly or nearly a maximum or minimum with respect to changes in w_a. In most such situations h_a will be small and h_{aa} will be important. It will then be possible to choose v_a so that, while g_{aa} does not vanish, $g_{aaa} = g_{aaaa} = \cdots = 0$. (Details are left to the reader.)

With this exception, then, the transformation will ordinarily be permissible, and we need only be concerned that the $v^{(1)}$-average point is not the same as the w-average point. If we did not know the location of the w-average point precisely, this fact will not concern us very much. Either we did not know the partial effect very well either, since it depends on the average point, and everything is really about equally hazy, or we were able to work with formulas in which the coordinates of the w-average point is expressed by letters which we may substitute for when we are done -- and nothing prevents us from making a substitution corresponding to the new average point, although (if we have exact knowledge) we may have to iterate a little to find the correct values for substitution. Thus the shift-in-average-point-problem is not likely to be important.

If we follow the program suggested above in detail, we would reach

$$z = g(v_1, v_2, \ldots, v_k)$$

with not only

$$g_{aa} = g_{aaa} = \ldots = 0$$

but also

$$g_a = 1$$

for all a. It is often natural to introduce particular v's which make the successive derivatives vanish, but do not have all $g_a = 1$. We begin, then, with this more general case.

8. Reduced Propagation Formulas

In the usual instances, we are merely, in system-component terms, planning to measure the performance of a component by the performance of a system containing this component but with all the other components selected to be at the mean values of each's distribution.

Under these conditions, the formulas simplify considerably. We now have $z = g(v_1, v_2, \ldots, v_k)$, where

(1) the v_a fluctuate independently,

(2) g_{aa}, g_{aaa}, etc. all vanish,

and the generalized propagation formulas reduce to those of Table 1. Notice that, although the first correction terms to the average, variance and skewness are all of order σ^4, their relative orders are σ^4, σ^2 and σ, respectively. Thus correction terms are most likely to have made a noticeable fractional change in the skewness, next in the variance, and least in the average. Note further, that, except in the leading term for the elongation, only the variances $\sigma_a{}^2$ and skewnesses $\gamma_a \sigma_a{}^3$ of the individual variables, the v_a, enter the generalized propagation formulas in terms of order $\leq \sigma^5$. Thus the requirements as to knowledge of component variability (<u>in the new terms</u>!) are not as stringent as we might have feared.

Table 1

Propagation in the Special Case Where g_{aa}, g_{aaa}, etc. Vanish

(Individual Variables measured proportionately to partial effects)

$$\text{ave } z = z_o \quad \text{(i.e. z calculated at the average point)}$$

$$+ \frac{1}{4} \Sigma^* g_{aabb} \sigma_a^2 \sigma_b^2$$

$$+ \frac{1}{12} \Sigma^* g_{aaabb} \gamma_a \sigma_a^3 \sigma_b^2$$

$$+ \text{ terms of order } \geq \sigma^6$$

$$\text{var } z = \Sigma g_a^2 \sigma_a^2 \quad \text{(i.e. the classical term)}$$

$$+ \Sigma^* (g_a g_{abb} + g_{ab}^2 + g_{aab} g_b) \sigma_a^2 \sigma_b^2$$

$$+ \Sigma^* (\frac{1}{2} g_a g_{aabb} + g_{ab} g_{aab} + \frac{1}{3} g_{aaab} g_b) \gamma_a \sigma_a^3 \sigma_b^2$$

$$+ \text{ terms of order } \geq \sigma^6$$

$$\text{ske } z = \Sigma g_a^3 \gamma_a \sigma_a^3$$

$$+ 6\Sigma^* g_a g_b g_{ab} \sigma_a^2 \sigma_b^2$$

$$+ 3\Sigma^* g_a g_{ab}^2 (\gamma_a - 1) \sigma_a^3 \sigma_b^2$$

$$+ \frac{3}{2} \Sigma^* (g_a^2 g_{abb} + 2 g_a g_{aab} g_b) \gamma_a \sigma_a^3 \sigma_b^2$$

$$+ \text{ terms of order } \geq \sigma^6$$

$$\text{elo } z = \Sigma g_a^4 (\Gamma_a - 3) \sigma_a^4$$

$$+ \text{ terms of order } \geq \sigma^6$$

(Symmetry of g_{ab} and g_{aabb} in a and b to be recognized in interpreting Σ^*)

We are now prepared to seek further condensation of these propagation formulas. We may gain a little by arranging for all the g_a to be unity -- we had already expressed the individual variables in terms such that g_{aa}, g_{aaa}, etc. all vanish, we can still choose the size of the units in terms of which they are expressed. We shall then be measuring each individual variable exactly in terms of its partial effect of its response. (This is entirely analogous to inspecting components by measuring the performance of a standard system in which <u>all other</u> components are at the average values of their distributions.)

If we introduce reduced or standardized quantities τ_a through

$$\tau_a = g_a \sigma_a ,$$

then τ_a^2 will be the variance of the reduced variable $x_a = g_a v_a$. Moreover, γ_a and Γ_a will be the same for the distribution of x_a as for that of v_a. We only need to define some t's by

$$t_{ab} = \frac{g_{ab}}{g_a g_b} , \quad t_{abbb} = \frac{g_{abbb}}{g_a g_b^3}$$

and the like, where the subscripts, as on t's and s's generally, do <u>not</u> mean simple differentiation, in order to obtain the propagation formulas in the form given in Table 2.

Table 2

The Propagation Formulas in Reduced Form

(Individual variables measured as partial effects)

$$\text{ave } z = z_o \quad \text{(i.e. } z \text{ evaluated at the average point)}$$

$$+ \Sigma^* \left(\frac{1}{4} t_{aabb}\right) \tau_a^{\,2} \tau_b^{\,2}$$

$$+ \Sigma^* \left(\frac{1}{12} t_{aaabb}\right) \gamma_a \tau_a^{\,3} \tau_b^{\,2}$$

$$+ \text{terms of order} \geq \tau^6$$

$$\text{var } z = \Sigma \tau_a^{\,2} \quad \text{(i.e. the classical term)}$$

$$+ \Sigma^* \left(t_{ab}^{\,2} + t_{aab} + t_{abb}\right) \tau_a^{\,2} \tau_b^{\,2}$$

$$+ \Sigma^* \left(\frac{1}{2} t_{aabb} + t_{ab} t_{aab} + \frac{1}{3} t_{aaab}\right) \gamma_a \tau_a^{\,3} \tau_b^{\,2}$$

$$+ \text{terms of order} \geq \tau^6$$

$$\text{ske } z = \Sigma \gamma_a \tau_a^{\,3} \quad \text{(i.e. the sum of component skewnesses)}$$

$$+ \Sigma^* (6\, t_{ab}) \tau_a^{\,2} \tau_b^{\,2}$$

$$+ \Sigma^* (3\, t_{ab}^{\,2})(\gamma_a - 1) \tau_a^{\,3} \tau_b^{\,2} + \Sigma^* (3 t_{aab} + \tfrac{3}{2} t_{abb}) \gamma_a \tau_a^{\,3} \tau_b^{\,2}$$

$$+ \text{terms of order} \geq \tau^6$$

$$\text{elo } z = \Sigma (\Gamma_a - 3) \tau_a^{\,4} \quad \text{(i.e. the sum of component elongations)}$$

$$+ \text{terms of order} \geq \tau^6$$

(Symmetry of t_{ab} and t_{aabb} in a and b to be recognized in interpreting Σ^*.)

In preparing Table 2 we have tried to throw each formula into its most usable form, and have consequently combined numerical coefficients with the t's they modify. In general, the most important terms, beyond the leading term in each formula, seem likely to be the second and third terms for the skewness and the second term for the variance.

9. The Causes and Justifications of Simplification

We have now reached a fairly compact set of propagation formulas which contain all terms through those of order τ^5. It is time for us to inquire into the assumptions, explicit and implicit, which have entered into the development-indicating the justification, complete or partial, of each. The list to be examined includes assumptions that:

(i) the functions considered are sufficiently differentiable to have convergent Taylor series,

(ii) the functions considered are, locally, sufficiently like polynomials to be reasonably well approximated by the first terms of their Taylor series,

(iii) the variables taken as individual (= component) variables have independent fluctuations (at least the cross-moments of order < 5 vanish when the variables are expressed in particular terms),

(iv) we may use an expansion around the average values of the individual (= component) variables, expressed in whatever terms we may use,

(v) the response (= overall) variable is a strictly monotone function of each individual (= component) variable.

The strength of (i) and (ii) can be fairly well discussed in terms of $z = h(w)$ where the function h is used as a rectifier. If we have a quadratic rectifier, $z = cw^2$, then the basic formulas (though not those reduced by transformation of individuals) will apply. If we have an ideal linear rectifier, $z = b|w|$, then (i) fails because of the misbehavior of the derivative at zero. If we observe that actual "linear" rectifiers deviate from 'ideal' behavior at zero, and manage to behave as if they have a continuous derivative, we find that (i) is satisfied, but that we are likely to be troubled with (ii), since the first derivative is nearly constant, except near zero, where it changes rapidly. This may well be far from polynomial behavior! Only if our 'linear' rectifier is quite far from 'ideal' can we expect even generalized propagation-of-error formulas to be a close approximation. The application of (i) and (ii) to other, more complex, situations tends to follow the pattern set in this example. If a single value of each first partial derivative is sufficient to guide even the roughest design (and sometimes when this is not true) both (i) and (ii) are likely to be well enough satisfied.

The classical formulas are frequently written, not only for independent variables

$$\text{var } y \sim \Sigma \left(\frac{\partial f}{\partial x_a}\right)^2 \text{ var } x_a$$

but also for the general case

$$\text{var } y \approx \Sigma \left(\frac{\partial f}{\partial x_a}\right)^2 \text{ var } x_a$$

$$+ 2\Sigma^* \left(\frac{\partial f}{\partial x_a}\right) \left(\frac{\partial f}{\partial x_b}\right) \text{cov } (x_a, x_b)$$

Why have we not followed the pattern in setting forth the generalization? Both because it would be difficult, and because it would really add little. The added complexity, for the classical formula, is small, and pays for itself in flexibility -- in the ability to work with intermediate variables whose variances and convariances have been already obtained by previous uses of propagation formulas -- in the ability to deal with situations where it is hard, or inconvenient, to find statistically independent variables. In practice it seems to be the first ability rather than the second which is usually important. Thus the availability of the classical formulas for the dependent case buys something, but not very much. Building provision for statistically dependent component variables into the generalized formulas would greatly complicate them -- so much so that it seems unlikely that terms beyond those of order τ^3 (or perhaps τ^4) would be written out. Thus the effort of seeking basic individual variables with independent fluctuations is likely to be far less than the effort of working with formulas for dependent ones. Thus (iii) has much practical justification. (In those instances where it does not hold, it will, of course, be necessary either to use the classical formulas or develop further approximations.)

What of the remaining assumption -- that we use an expansion around the average values of our individual variables? When we recognize the inaccuracies of our knowledge in any practical application, this assumption does not really seem serious. If our knowledge is basically mathematical and approximate, gained by calculating values of partial derivatives at a certain point, then it may be true that this point would not be the average point in a real situation. But it is quite likely to be true that we do not know accurately where the real average point will be. This means that we must interpret our derivatives, and all that hangs upon them, with caution and some allowance for variation. This will be the case for any formula, whether or not it expands around the average point.

In order to deal wisely with this possibility, we need to know how the g_a and t_{ab} change when we move from the point $v_{10}, v_{20}, \ldots, v_{k0}$ to the nearby point $v_{11}, v_{21}, \ldots, v_{k1}$. If we let $v_1, v_2, \ldots, v_k$ be the displacements, as measured by the response at the average point, i.e.

$$v_a = g(v_{10}, v_{20}, \ldots, v_{a1}, \ldots, v_{k0}) - g(v_{10}, v_{20}, \ldots, v_{a0}, \ldots, v_{k0})$$

then the original forms and first order correction terms are indicated by the following expressions

$$g_a(1 + \Sigma_c\, t_{ac}\, \nu_c + \ldots)$$

$$t_{ab}(1 + \Sigma_c(t_{abc} - t_{ac} - t_{bc})\, \nu_c + \ldots)$$

$$t_{aab}(1 + \Sigma_c(t_{aabc} - 2\, t_{ac} - t_{bc})\nu_c + \ldots)$$

and so on, including

$$t_{aaabb}(1 - \Sigma_c(t_{aaabbc} - 3\, t_{ac} - 2\, t_{bc})\nu_c + \ldots) .$$

These formulas provide the necessary guide, should it ever be necessary to quantitatively judge the adequacy of the approximation with which (iv) holds. We may expect that it will usually be good.

III EXAMPLES

10. The First Example

We now report the results of quantitative discussion of two moderately simple examples beginning with the time delay of a lumped-constant delay line, each section of which consists of an inductance L and capacitance C (initially) of the same nominal values. (The detailed calculations may be found in Part VI.) The original functional relation may be written

$$\text{delay} = z = \sqrt{L_1 C_2} + \sqrt{L_3 C_4} + \ldots + \sqrt{L_{2j-1} C_{2j}}$$

where we have used odd subscripts on inductances and even ones on capacitances so that a subscript points uniquely to a component.

For convenient insight, we fix the units for L and C so that their nominal values are both 10. This means that $\sigma_a = 1$ corresponds to a standard deviation of 10% in the inductance or capacity of component a. (Realistically, then, σ's may be as large as 2, but are more likely to be fractions like 0.6, 0.2 or 0.06.)

Direct application of the original formulas now gives (terms arranged in order of size of coefficient -- those with coefficients smaller than 0.000005 omitted)

$$\text{ave } z = 10j - 0.0125\, \Sigma\, \sigma_a^2 + 0.00062\, \Sigma\, \gamma_a \sigma_a^3$$

$$-0.00002\, \Sigma (2\Gamma_{2i-1}\sigma_{2i-1}^4 - \sigma_{2i-1}^2 \sigma_{2i}^2 + 2\Gamma_{2i}\sigma_{2i}^4)$$

$$+ \ldots$$

$$\begin{aligned}\text{var } z = {} & 0.25\, \Sigma\, \sigma_a^{\,2} - 0.01250\, \Sigma\, \gamma_a \sigma_a^{\,3} \\ & + 0.00016\, \Sigma \left[(5\Gamma_{2i-1} - 1)\sigma_{2i-1}^{\;4} - 4\sigma_{2i-1}^{\;2}\sigma_{2i}^{\;2} + (5\Gamma_{2i} - 1)\sigma_{2i}^{\;4} \right] \\ & + 0.00002\, \Sigma (G_a - \gamma_a)\sigma_a^{\,5} - 0.00003\, \Sigma (\gamma_{2i-1}\sigma_{2i-1} + \gamma_{2i}\sigma_{2i})\sigma_{2i-1}^{\;2}\sigma_{2i}^{\;2} \\ & + \ldots \end{aligned}$$

$$\begin{aligned}\text{ske } z = {} & 0.125\, \Sigma\, \gamma_a \sigma_a^{\,3} \\ & - 0.00937\, \Sigma \left[(\Gamma_{2i-1} - 1)\sigma_{2i-1}^{\;4} - 4\sigma_{2i-1}^{\;2}\sigma_{2i}^{\;2} + (\Gamma_{2i} - 1)\sigma_{2i}^{\;4} \right] \\ & + 0.00023\, \Sigma\, (3G_a - 4\gamma_a)\sigma_a^{\,5} \\ & - 0.00047\, \Sigma \left[(3\gamma_{2i-1} + 2)\sigma_{2i-1} + (3\gamma_{2i} + 2)\sigma_{2i} \right] \sigma_{2i-1}^{\;2}\sigma_{2i}^{\;2} \\ & + \ldots \end{aligned}$$

$$\begin{aligned}\text{elo } z = {} & 0.0625\, \Sigma\, (\Gamma_a - 3)\sigma_a^{\,4} + 0.00625\, \Sigma\, (G_a - 4\gamma_a)\sigma_a^{\,5} \\ & + \ldots \end{aligned}$$

and it is clear that, in every case, at most the first correction term will have any numerical importance.

If we transform the individual variables, to make partial effects linear of coefficient one, then we have

$$\tau_a = \frac{1}{2}\, \sigma_a \, ,$$

where σ_a is the standard deviation of the new a^{th} variable, and

$$\text{ave } z = 10j + \text{no other terms}$$

$$\text{var } z = \Sigma \tau_a^2 + 0.01 \Sigma \tau_{2i-1}^2 \tau_{2i}^2 + \text{no other terms,}$$

$$\text{ske } z = \Sigma \gamma_a \tau_a^3 + 0.6 \Sigma \tau_{2i-1}^2 \tau_{2i}^2$$

$$+ 0.03 \Sigma \left[(\gamma_{2i-1} - 1)\tau_{2i-1} + (\gamma_{2i} - 1)\tau_{2i} \right] \tau_{2i-1}^2 \tau_{2i}^2$$

$$+ \text{two sets of terms of order } \tau^6$$

$$\text{elo } z = \Sigma(\Gamma_a - 3)\tau_a^4 + \text{terms of order from } \tau^6 \text{ to } \tau^8,$$

where subscripts a run from 1 to 2j and subscripts i from 1 to j. The qualitative simplification is obvious. One term instead of 5 plus higher terms in the average. Two instead of five plus higher in the variance. Three plus two higher instead of four plus higher in the skewness. One plus higher instead of two plus higher in the elongation. The quantitative simplification in the variance is also important. If $\sigma_a \leq 1$ for all a, the second term in the variance is less than one-thousandth of the first.

In the skewness, however, the second term retains considerable importance in comparison with the first. Practical use of these formulas would probably involve one term each for average, variance, and, if it were required, elongation. If the skewness were required, at least the second term would, and the third term might, need examination. In terms of transformed individual variables, then, terms beyond classical propagation of error need be considered in at most one formula.

11. The first example generalized.

If we now consider a delay line in which the nominal values of the elements need not be the same from section to section, it becomes convenient to introduce new quantities

$$t_i = \sqrt{L_{2i-1}\ C_{2i}} = \text{time delay of ith section.}$$

$$\eta_{2i-1} = \frac{\text{standard deviation of } \sqrt{L_{2i-1}}}{\text{average value of } \sqrt{L_{2i-1}}} = \frac{\tau_{2i-1}}{t_i},$$

$$\eta_{2i} = \frac{\text{standard deviation of } \sqrt{C_{2i}}}{\text{average value of } \sqrt{C_{2i}}} = \frac{\tau_{2i}}{t_i}.$$

In terms of these quantities, the formulas become

$$\text{ave } z = \Sigma t_i,$$

$$\text{var } z = \Sigma t_i^{2}(\eta_{2i-1}^{2} + \eta_{2i}^{2} + \eta_{2i-1}^{2}\ \eta_{2i}^{2}),$$

$$\text{ske } z = \Sigma t_i^{3}\Big[\gamma_{2i-1}\eta_{2i-1}^{3} + \gamma_{2i}\eta_{2i}^{3} + 6\eta_{2i-1}^{2}\ \eta_{2i}^{2}$$

$$+ 3(\gamma_{2i-1}-1)\eta_{2i-1}^{3}\eta_{2i}^{2} + 3(\gamma_{2i}-1)\eta_{2i}^{3}\eta_{2i-1}^{2})\Big]$$

$$+ \text{terms of order} \geq \tau^{6},$$

$$\text{elo } z = \Sigma t_i^{4}\Big[(\Gamma_{2i-1}-3)\eta_{2i-1}^{4} + (\Gamma_{2i}-3)\eta_{2i}^{4}\Big]$$

$$+ \text{terms of order} \geq \tau^{6},$$

and are clearly easily manageable.

12. The second example.

The attenuation caused by a π-section network of series resistance R_B and shunt resistance R_A and R_C, operating between image impedances R, is

$$z = 1 + \frac{1}{2R} R_B + \frac{1}{2} R_B \left\{\frac{1}{R_A} + \frac{1}{R_C}\right\} + \frac{R}{2}\left\{\frac{1}{R_A} + \frac{1}{R_C} + \frac{R_B}{R_A R_C}\right\}.$$

If we write this in terms of changes from nominal, the unit for each change being 10% of the corresponding nominal value, we find

$$R_A = R\,\frac{\alpha - 1}{\alpha + 1}\,(1 + 0.1\ w_A)$$

$$R_B = R\,\frac{\alpha^2 - 1}{2\alpha}\,(1 + 0.1\ w_B)$$

$$R_C = R\,\frac{\alpha - 1}{\alpha + 1}\,(1 + 0.1\ w_C)$$

where α is the design attenuation, and calculate the necessary derivatives, the generalized propagation formulas become, if we introduce abbreviations β, γ (not the same as γ_A, γ_B, γ_C for the w's) and δ by

$$\beta = \frac{\alpha}{2}\,\frac{\alpha - 1}{\alpha + 1}, \quad \gamma = \frac{(\alpha-1)^3}{4\alpha(\alpha + 1)}, \quad \delta = \frac{(\alpha - 1)^2}{2(\alpha + 1)},$$

the following:

$$\text{ave } z = a \quad \text{(provided the averages of } w_A, w_B \text{ and } w_C \text{ vanish)}$$

$$+ 0.01\ \beta(\sigma_A{}^2 + \sigma_C{}^2)$$

$$- 0.001\ \beta(\gamma_A\ \sigma_A{}^3 + \gamma_C\sigma_C{}^3)$$

$$+ 0.0001\ \beta(\Gamma_A\sigma_A{}^4 + \Gamma_C\sigma_C{}^4)$$

$$+ 0.0001\ \gamma\sigma_A{}^2\sigma_C{}^2$$

$$- 0.00001\ \beta(G_A\sigma_A{}^5 + G_C\sigma_C{}^5)$$

$$+ 0.00001\ \gamma\ (\gamma_A\sigma_A + \gamma_C\sigma_C)\ \sigma_A{}^2\sigma_C{}^2$$

$$+ \ldots$$

$$\text{var } z = 0.01\ \beta^2\ (\sigma_A{}^2 + 4\sigma_B{}^2 + \sigma_C{}^2)$$

$$- 0.002\ \beta^2\ (\gamma_A\sigma_A{}^3 + \gamma_C\sigma_C{}^3)$$

$$+ 0.0001\ \beta^2 \left[(3\Gamma_A - 1)\ \sigma_A{}^4 + (3\Gamma_C - 1)\sigma_C{}^4\right]$$

$$+ 0.0001 \left[(\delta^2 + 4\beta\delta)(\sigma_A{}^2 + \sigma_C{}^2)\sigma_B{}^2 + (\gamma^2 + 4\beta\gamma)\sigma_A{}^2\sigma_C{}^2\right]$$

$$- 0.00002\ \beta^2 \left[(2G_A - \gamma_A)\sigma_A{}^5 + (2G_C - \gamma_C)\ \sigma_C{}^5\right]$$

(formula continues)

$$- 0.00001 \left[(2\delta^2 + 4\beta\delta)(\gamma_A \sigma_A^3 + \gamma_C \sigma_C^3)\sigma_B^2 \right.$$

$$\left. -(2\gamma^2 + 3\beta\gamma)(\gamma_A \sigma_A + \gamma_C \sigma_C)\ \sigma_A^2 \sigma_C^2 \right]$$

$$+ \ldots$$

ke $z = - 0.001\ \beta^3\ (\gamma_A \sigma_A^3 - 8\gamma_B \sigma_B^3 + \gamma_C \sigma_C^3)$

$$- 0.0003\ \beta^3 \left[(\Gamma_A - 1)\ \sigma_A^4 + (\Gamma_C - 1)\ \sigma_C^4 \right]$$

$$+ 0.0012\ \beta^2\delta\ (\sigma_A^2 + \sigma_C^2)\ \sigma_B^2 + 0.0006\ \beta^2\gamma\ \sigma_A^2 \sigma_C^2$$

$$- 0.00003\ \beta^3 \left[(2G_A - 3\gamma_A)\ \sigma_A^5 + (2G_C - 3\gamma_C)\ \sigma_C^5 \right]$$

$$- 0.00003\ \beta\delta^2 \left[(\gamma_A - 1)\sigma_A^3 \sigma_B^2 - 2(\gamma_B - 1)\sigma_B^3(\sigma_A^2 + \sigma_C^2) \right.$$

$$\left. + (\gamma_C - 1)\sigma_C^3 \sigma_B^2 \right]$$

$$- 0.00003\ \beta\gamma^2 \left[(\gamma_A - 1)\sigma_A + (\gamma_C - 1)\sigma_C \right] \sigma_A^2 \sigma_C^2$$

$$- 0.00006\ \beta^2\delta \left[(4\gamma_A \sigma_A^3 + 4\gamma_C \sigma_C^3)\sigma_B^2\ \ 2\gamma_B \sigma_B^3 (\sigma_A^2 + \sigma_C^2) \right]$$

$$- 0.00015\ (\gamma_A \sigma_A + \gamma_C \sigma_C)\ \sigma_A^2 \sigma_C^2$$

$$+ \ldots$$

$$\text{elo } z = 0.0001\,\beta^4\left[(\Gamma_A - 3)\sigma_A^{\,4} + 16(\Gamma_B - 3)\,\sigma_B^{\,4} + (\Gamma_C - 3)\sigma_C^{\,4}\right]$$

$$- 0.00004\,\beta^4\left[(G_A - 4\gamma_A)\,\sigma_A^{\,5} + (G_C - 4\gamma_C)\,\sigma_C^{\,5}\right]$$

$$+ \ldots$$

While some of these formulas appear rather lengthy, it is clear that the first two terms will give either the average or the variance to working accuracy.

If we seek to transform the individual variables, we are immediately led to begin by replacing the resistances R of the shunt arms by their conductances Y. The attenuation then becomes

$$z = 1 + \frac{1}{2R} R_B + \frac{1}{2} R_B \left\{Y_A + Y_C\right\} + \frac{R}{2}\left\{Y_A + Y_C + R_B Y_A Y_C\right\}$$

and we place

$$Y_A = \frac{1}{R}\ \frac{\alpha - 1}{\alpha + 1}\ (1 + 0.1\ v_A)$$

$$R_B = R\ \frac{\alpha^2 - 1}{2\alpha}\ (1 + 0.1\ v_B)$$

$$Y_C = \frac{1}{R}\,\frac{\alpha - 1}{\alpha + 1}\ (1 + 0.1\ v_C)$$

(Note that $dW_A/dV_A) = (dW_C/dV_C) = -1$, not +1. This deviation from our general practice is a matter of convenience only.)

The generalized propagation formulas now become:

ave $z = \alpha + \ldots$ (provided the average values of v_A, v_B and v_C vanish)

$$\text{var } z = 0.01\, \beta^2 (\sigma_A^2 + 4\sigma_B^2 + \sigma_C^2) + 0.0001\, \delta^2(\sigma_A^2 + \sigma_C^2)\, \sigma_B^2 + 0.0001\, \gamma^2\, \sigma_A^2\, \sigma_C^2 + \ldots$$

$$\begin{aligned} \text{ske } z = {} & 0.001\, \beta^3 (\gamma_A \sigma_A^3 + 8\gamma_B \sigma_B^3 + \gamma_C \sigma_C^3) \\ & + 0.0012\, \beta^2 \delta\, (\sigma_A^2 + \sigma_C^2)\, \sigma_B^2 + 0.0006\, \beta^2 \gamma\, \sigma_A^2 \sigma_C^2 \\ & + 0.00003\, \beta\delta^2 \left[(\gamma_A - 1)\sigma_A^3 + (\gamma_C - 1)\, \sigma_C^3 \right] \sigma_B^2 \\ & + 0.00003\, \beta\gamma^2 \left[(\gamma_A - 1)\sigma_A + (\gamma_C - 1)\sigma_C \right] \sigma_A^2 \sigma_C^2 \\ & + 0.00006\, \beta\delta^2\, (\gamma_B - 1)\, \sigma_B^3\, (\sigma_A^2 + \sigma_C^2) \\ & + \ldots \end{aligned}$$

$$\text{elo } z = 0.0001\, \beta^4 \left[(\Gamma_A - 3)\sigma_A^4 + 16(\Gamma_B - 3)\sigma_B^4 + (\Gamma_C - 3)\sigma_C^4 \right] + \ldots$$

While the second terms for the average and the variance might be worth computing roughly, they are unlikely to make enough of a contribution to bother with.

13. Numerical Examples

Let us consider the second example, and suppose that the elements to be used are 5% resistors, 10% resistors or 20% resistors (i.e. have these limits on deviations from their nominal values) and let us, for simplicity, assume that their distributions (within these limits), are uniform. Then the necessary numerical values will be as in Table 3, which also gives the formulas for the cumulants of the attenuation in terms of σ^2 for a nominal attenuation of 10. These formulas will converge most slowly when all resistors are 20% resistors. In this situation, combining terms of the same order,

$$\text{ave } z = 10.0000 + 0.1091 + 0.0029 + \ldots = 10.1120 + \ldots$$

$$\text{var } z = 1.3389 + 0.0524 + \ldots = 1.3913 + \ldots$$

$$\text{ske } z = 0.0000 + 0.2341 + \ldots = 0.2341 + \ldots$$

$$\text{elo } z = -1.0755 + \ldots$$

Table 3

Numerical quantities for uniformly distributed resistances within specification

Resistor	Limits on W	Limits on w	σ^2 for w	γ	Γ	G
5%	±.05	±.5	.0833	0	1.8	0
10%	±.10	± 1	.3333	0	1.8	0
20%	±.20	± 2	1.3333	0	1.8	0

Formulas for cumulants of attenuation in terms of σ^2 (through terms in σ^4)

$$\text{ave } z = 10.00000 + 0.04091(\sigma_A^2+\sigma_C^2)+0.00074(\sigma_A^4+\sigma_C^4) + .00017\,\sigma_A^2\sigma_C^2 + \dots$$

$$\text{var } z = 0.16736(\sigma_A^2 + 4\sigma_B^2+\sigma_C^2)+0.00736(\sigma_A^4+\sigma_C^4)$$

+0.0033[illegible] $(\sigma_A^2+\sigma_C^2)\sigma_B^2$ + 0.00[illegible]39 $\sigma_A^2\sigma_C^2$ + ...

$$\text{ske } z = -0.01642(\sigma_A^4+\sigma_C^4)+0.07381(\sigma_A^2+\sigma_C^2)\sigma_B^2+0.01688\sigma_A^2\sigma_C^2 + \dots$$

$$\text{elo } z = -0.03368(\sigma_A^4 + 16\sigma_B^4+\sigma_C^4)+\dots$$

- -

Clearly the leading terms in the average, variance and elongation need little adjustment. The first term in the skewness is zero, so that the leading correction term contributes substantially. There is no reason to believe that the neglected terms in σ^5 and higher will affect more than the 6th significant figure in the average or the 4th in the variance. If this be so, then unconsidered terms are utterly negligible.

The treatment just given assumed a uniform distribution of resistance within tolerances. This is not likely to be the case with any actual supply of resistors. In practice it is not unlikely that we should know enough about the actual distribution to make a somewhat more reasonable assumption but our knowledge will always be far from perfect. The assumption that the distributions of conductances for the shunt elements, and the distribution of resistance for the series element, are all uniform within tolerances is manifestly arbitrary. But it is not a bit more arbitrary than the previous assumption. (We might expect the logarithm of resistance or of conductance to be nearly uniform under certain conditions.) If we make this assumption, then we have to deal with the numbers and formulas of Table 4. Here, if $\alpha = 10$ and all resistors are 20% resistors:

$$\text{ave } z = 10.0000 + \ldots = 10.0000 + \ldots$$

$$\text{var } z = 1.3389 + 0.0048 + \ldots = 1.3437 + \ldots$$

$$\text{ske } z = 0.0000 + 0.2925 + \ldots = 0.2925 + \ldots$$

$$\text{elo } z = -1.07547 + \ldots$$

Table 4

Numerical quantities for uniform distribution within tolerances for shunt conductances and series resistances

Resistor	Limits on V	Limits on v	σ^2 for v	for v γ	or V Γ	G
5%	±0.05	±.5	0.0833	0	1.8	0
10%	±0.1	± 1	0.3333	0	1.8	0
20%	±0.2	± 2	1.333	0	1.8	0

Formulas for cumulants of attenuation in terms of σ^2 (of v's) when $\alpha = 10$

$$\text{ave } z = 10.00000 + \ldots$$

$$\text{var } z = 0.16736(\sigma_A^2+4\sigma_B^2+\sigma_C^2) + 0.00136(\sigma_A^2+\sigma_C^2)\,\sigma_B^2 + \ldots$$

$$\text{ske } z = 0.07394\,(\sigma_A^2+\sigma_C^2)\sigma_B^2 + 0.01664\,\sigma_A^2\sigma_C^2 + \ldots$$

$$\text{elo } z = -\,0.03361(\sigma_A^4+16\sigma_B^4+\sigma_C^4) + \ldots$$

- -

The propagated values of average and variance are somewhat different from the results obtained under the previous assumption. These differences must be slight from a practical point of view since they derive from assumptions which are practically indistinguishable. Practical accuracy will not be understated if we predict

$$\text{ave } z = 10.0, \qquad \text{ske } z = 0.3,$$

$$\text{var } z = 1.35, \qquad \text{elo } z = -1.1$$

for the result of using 20% resistors for all three elements.

Qualitatively, the indications are that one correction term for the skewness and no correction terms for the other cumulants (average, variance and elongation) will suffice if we work with suitably transformed individual variables.

14. Probabilities of deviations beyond tolerances

Let us continue the previous example, supposing that attenuations between 8.5 and 12.0 are acceptable, while those outside these limits are not acceptable. Let us begin by calculating probabilities by the normal approximation. For all resistors 20%, ave z = 10.12 and var z = 1.34, so that the limits lie at standardized deviates of

$$\frac{8.5 - 10.0}{\sqrt{1.35}} = -1.29 \quad \text{and} \quad \frac{12.0 - 10.0}{\sqrt{1.35}} = 1.72$$

respectively. Reference to a table of the standard normal distribution indicates probabilities of 9.85% for too low attenuation, and 4.27% for too high attenuation.

If we wish to make better approximations we find first that

$$\gamma_1 = \frac{0.3}{(1.35)^{3/2}} = 0.19$$

$$\gamma_2 = \frac{-1.1}{(1.35)^2} = -0.60$$

and, consequently, $\beta_1 = 0.036$, $\beta_2 = 2.40$. Reference to the Merrington-Pearson table (specifically table 42 in Pearson and Hartley) for these values of β_1 and β_2 (and positive skewness) leads to the following

%	Lower Deviate Normal	.036,2.40	Diff.	Upper Deviate Normal	.017, 2.78	Diff.
5%	-1.645	-1.57	$+.07_5$	+1.645	1.72	$+.07_5$
2.5%	-1.960	-1.76	$+.20_0$	+1.960	1.99	$+.03_0$
1%	-2.325	-1.93	$+.39_5$	+2.325	2.26	$-.06_5$

We are thus led to modify our standardized deviates as follows

$$-1.29 + 0.01 = -1.28$$

$$+1.72 + 0.06 = 1.78$$

These modified deviates, when looked up in a unit normal table correspond to 10.03% below and 3.75% above, the changes in the two tails tending to compensate one another. The total % outside tolerance decreases by 0.52% - 0.18% = 0.34% out of about 14%. Clearly this change has little practical significance.

Thus, as illustrated above, the methods of using higher cumulants are often of most value when they demonstrate the unimportance of the correction.

TRANSFORMATION OF TERMS OF RESPONSE

15. General formulae

We have now to consider the possibilities of, and necessary machinery for, changes in the terms in which the response is measured. We shall clearly wish to keep partial effects linear. This requires us to change the terms of the individual variables at the same time that we change the terms of response.

If $z = g(v_1, v_2, \ldots, v_k)$ was such that g_{aa}, g_{aaa}, etc. all vanished, we want to apply $y = \varphi(z)$ and compensate for this by also applying $v_a = v_a(u_a)$, so that in the resulting

$$y = f(u_1, u_2, \ldots, u_k)$$

we shall have all f_{aa}, f_{aaa}, etc. vanish. If we maintain the verbal definition of the τ_a — the standard deviation of the partial effect of individual variable a — rather than maintaining its numerical values, so that now

$$\tau_a^2 = f_a^2 \text{ var } u_a,$$

then the propagation formulas involve the τ_a and the reduced higher (cross) derivative values of f, defined by

$$s_{ab} = \frac{f_{ab}}{f_a f_b}, \quad s_{aabbb} = \frac{f_{aabbb}}{f_a^2 f_b^3}$$

and the like. We can, fortunately, express the s's rather simply in terms of the t's and the derivative values of φ,

$$\varphi' = \left(\frac{d\varphi}{dz}\right)_{(\text{ave } z)}, \qquad \varphi'' = \left(\frac{d^2\varphi}{dz^2}\right)_{(\text{ave } z)}, \quad \ldots$$

The resulting formulas, derived in part VIII, are

$$s_{ab} = t_{ab} + \varphi''$$

$$s_{abb} = t_{abb} + \varphi'' t_{ab} + \varphi''' - \varphi''\varphi'',$$

$$s_{abbb} = t_{abbb} + (2\varphi''' - 3\varphi''\varphi'')\, t_{ab} + \varphi^{iv} - 4\varphi'''\varphi'' + 3\varphi''\varphi''\varphi'',$$

$$s_{aabb} = t_{aabb} + \varphi''\,(t_{aab} + 2\, t_{ab}{}^{2} + t_{abb})$$
$$+ (4\varphi''' - 3\varphi''\varphi'')\, t_{ab} + \varphi^{iv} - 2\varphi'''\varphi'' + \varphi''\varphi''\varphi''$$

and, indeed,

$$s_{aaabb} = t_{aaabb} + 6\varphi''\, t_{ab} t_{aab} + 6(\varphi''' - \varphi''\varphi'')\, t_{aab}$$
$$+ (2\varphi''' - 3\varphi''\varphi'')\, t_{abb} + 6(\varphi''' - \varphi''\varphi'')\, t_{ab}{}^{2}$$
$$+ (6\varphi^{iv} - 16\varphi'''\varphi'' + 9\varphi''\varphi''\varphi'')\, t_{ab}$$
$$+ \varphi^{v} - 4\varphi^{iv}\varphi'' - \varphi'''\varphi''' + 7\varphi'''\varphi''\varphi'' - 3\varphi''\varphi''\varphi''\varphi''$$

16. Specializations for power transformations

We are usually not going to use a more complicated transformation than $y = A(z+c)^p$ (or one of the limiting forms of this family, which include logarithmic and exponential forms). The higher derivative values, φ^{iv}, φ^{v} can be thrown back onto φ', φ'' and φ''' for any such power transformation (see Part VIII for details) and if we fix $\varphi' = 1$ and use these relations, we may find formulas for directly changing the actual coefficients of Table 2. These formulas are given in Table 5. The abbreviations

$$D = \varphi''' - \varphi''\varphi''$$

and

$$E = \varphi''\varphi''$$

contribute substantially to compactness.

TABLE 5

Effect on coefficients of a power transformation (with $\varphi' = 1$) of response and related transformations of the individual variables.

Abbreviations: $D = \varphi''' - \varphi''\varphi''$, $E = \varphi''\varphi''$

$$s_{aabb} = t_{aabb} + \varphi''(t_{aab} + 2t_{ab}^2 + t_{abb}) + (4D+E)t_{ab} + \frac{1}{\varphi''} D(2D+E) ,$$

$$s_{aaabb} = t_{aaabb} + 6\varphi'' \, t_{ab} \, t_{aab} + 6D \, t_{aab} + (2D-E) \, t_{abb} + 6D \, t_{ab}^2 + \frac{1}{\varphi''} (12D^2 + 2DE - E^2) \, t_{ab} + \frac{1}{E} (6D^3 + 2D^2E - DE^2) ,$$

$$s_{aab} + s_{ab}^2 + s_{abb} = t_{aab} + t_{ab}^2 + t_{abb} + 4\varphi'' \, t_{ab} + (2D+E) ,$$

$$\frac{1}{2} s_{aabb} + s_{ab} \, s_{aab} + \frac{1}{3} s_{aaab} = \frac{1}{2} t_{aabb} + t_{ab} \, t_{aab} + \frac{1}{3} t_{aaab} + \varphi'' \left(\frac{3}{2} t_{aab} + 2t_{ab}^2 + t_{abb}\right) + \frac{1}{6} (22D+7E) \, t_{ab} + \frac{1}{6\varphi''} (10D^2 + 7DE)$$

$$s_{ab} = t_{ab} + \varphi''$$

$$s_{ab}^2 = t_{ab}^2 + 2\varphi'' \, t_{ab} + E ,$$

$$3s_{aab} + \frac{3}{2} s_{abb} = 3t_{aab} + \frac{3}{2} t_{abb} + \frac{9}{2} \varphi'' \, t_{ab} + \frac{9}{2} D$$

One substantial advantage of Table 5 is that it allows us to calculate the effect of a power transformation with chosen φ'' and φ''' on a particular set of propagation formulas without any necessity of calculating the power transformation explicitly. This often suffices, by showing the most that can be done.

Actually, as is shown by one of the examples (see Section 37), the apparent form of the power transformation is sometimes shifted very much by small changes in φ'' and φ'''. The nature of $y = \varphi(z)$ near $z = z_o$ will not change much, but the analytic form may change considerably — as from

$$y = (400)^{1/3}\ (z + \frac{10}{3})^{1/3}$$

to

$$y = -\ 213.7(z+14.42)^{-.2757}$$

which, apart from an additive constant, are quite similar near $z = 0$. The c in $(z + c)^p$ imparts more flexibility for a fixed p than one naturally supposes.

If one wishes to restrict attention to the simple power transformation $y = Az^p$, then only φ'' may be chosen, and

$$\varphi''' = \varphi''\varphi'' - \frac{1}{z_o}\varphi''\ ,\ D = -\ \varphi''/z_o,\ E = \varphi''\varphi''$$

are an inevitable consequence.

17. Modes of use

We may use this freedom of transformation of terms of response in various ways, the most promising of which seem to be either further simplification of formulae or reduction in skewness of response. More detailed inquiry in Part VIII leads us to the views (i) that the latter will only infrequently be practical, since its effective accomplishment depends on detailed quantitative knowledge of the γ_a and τ_a and (ii) that effort on simplification of formula is best concentrated on the annulling, or reduction in size, of the coefficient of $\tau_a^2 \tau_b^2$ in the variance.

Since this coefficient is, for any particular (a,b)

$$s_{aab} + s_{ab}^2 + s_{abb} = t_{aab} + t_{ab}^2 + t_{abb} + 4\varphi'' t_{ab} + 2\varphi''' - \varphi''\varphi''$$

and since we have both φ'' and φ''' at our disposal, we can exactly annul two different coefficients by a proper choice of φ. (How effective such a transformation may be will depend very much on the particular example.)

The examples considered later on indicate that the apparently simpler, and certainly less justified, approach of trying to annul some of the s_{ab} alone is not likely to be rewarding. If we are to gain in simplicity of formula by transforming the response, we must consider third derivatives as well as second derivatives so as to obtain and use t_{aab} and t_{abb}.

18. The examples

If we consider transforming the response in the first example, we find that we can eliminate the main correction term in the formula for the variance. Instead of

$$\text{var } z \approx \Sigma\, \tau_a^{\,2} + 0.01\, \Sigma\, \tau_{2i-1}^{\,2}\, \tau_{2i}^{\,2}$$

we obtain

$$\text{var } z \approx \Sigma\, \tau_a^{\,2} + 0.0005\, \Sigma(\gamma_{2i-1}\tau_{2i-1}+\gamma_{2i}\tau_{2i})\tau_{2i-1}^{\,2}\, \tau_{2i}^{\,2} \ldots +$$

If we had needed to reduce this term, the gain would be striking, even though the simplest transformation uses the cube of the difference between the time delay and a constant larger than any likely delay.

Transformation of the response in the second example again reduces the first correction term in the variance by a factor of 50 or so, but again this reduction was not badly needed. The power transformation which does this is proportional to

$$\sqrt[3]{z + \frac{\alpha}{3}}$$

interestingly enough.

V DETAILS OF PROPAGATION

19. Cocumulants – definitions and properties.

The covariance, defined by

$$\text{cov }(x,y) = \text{ave } xy - \text{ave } x \text{ ave } y = \text{ave } (x-\bar{x})(y-\bar{y})$$

and satisfying

(i) $\text{var } x+y = \text{var } x + 2 \text{ cov } (x,y) + \text{var } y$

(ii) $\text{var } x = \text{cov } (x,x)$

(iii) $\text{cov } (x,y) = \text{cov } (y,x)$

(iv) if y is (statistically) independent of x, then $\text{cov } (x,y) = 0$

(v) $\text{var } \Sigma x_a = \Sigma_a \Sigma_b \text{ cov } (x_a, x_b)$

(vi) $\text{cov } (\Sigma x_a, \Sigma y_b) = \Sigma_a \Sigma_b \text{ cov } (x_a, y_b)$

(vii) if v is independent of x, y together, then $\text{cov } (x, yv) = \text{cov } (x,y) \text{ ave } v$

has stood beside the variance for a long time. Similar partners for the higher cumulants do not seem to have been provided, though they will prove very convenient to us.

We may define the coskewness, a function of three arguments, by

$$\text{cok } (x,y,z) = \text{ave } (x-\bar{x})(y-\bar{y})(z-\bar{z})$$

(we use "cok" because "cos" is preempted) and can show directly that

(i) ske $x+y$ = ske x + 3 cok (x,x,y) + 3 cok (x,y,y) + ske y ,

(ii) ske x = cok (x,x,x) ,

(iii) cok (x,y,z) = cok (x,z,y) = ... = cok (z,y,x) ,

(iv) if z is (statistically) independent of x,y, together, then cok $(x,y,z) = 0$

(v) ske $\Sigma x_a = \Sigma_a \Sigma_b \Sigma_c$ cok (x_a, x_b, x_c)

(vi) cok $(\Sigma x_a, \Sigma y_b, \Sigma z_c) = \Sigma_a \Sigma_b \Sigma_c$ cok (x_a, y_b, z_c)

(vii) if v is independent of x,y,z together, then

$$\text{cok}\ (x,y,zv) = \text{cok}\ (x,y,z)\ \text{ave}\ v$$

in entire analogy with the properties of the covariance.

Moreover, we can define the <u>coelongation,</u> a function of 4 arguments, by

$$\begin{aligned} \text{coe}\ (x,y,z,w) = {} & \text{ave}\ (x-\bar{x})(y-\bar{y})(z-\bar{z})(w-\bar{w}) \\ & - \text{cov}\ (x,y)\ \text{cov}\ (z,w) \\ & - \text{cov}\ (x,z)\ \text{cov}\ (y,w) \\ & - \text{cov}\ (x,w)\ \text{cov}\ (y,z) \end{aligned}$$

and can show directly that

(i) $\text{elo } x+y = \text{elo } x + 4 \text{ coe } (x,x,x,y) + 6 \text{ coe } (x,x,y,y) + 4 \text{ coe } (x,y,y,y) + \text{elo } y$

(ii) $\text{elo } x = \text{coe } (x,x,x,x)$

(iii) $\text{coe } (x,y,z,w) = \text{coe } (x,y,w,z) = \ldots = \text{coe } (w,z,y,x)$

(iv) if w is (statistically) independent of x,y,z together, then $\text{coe } (x,y,z,w) = 0$

(v) $\text{elo } \Sigma x_a = \Sigma_a \Sigma_b \Sigma_c \Sigma_d \text{ coe } (x_a, x_b, x_c, x_d)$

(vi) $\text{coe } (\Sigma x_a, \Sigma y_b, \Sigma z_c, \Sigma w_d) = \Sigma_a \Sigma_b \Sigma_c \Sigma_d \text{ coe } (x_a, y_b, z_c, w_d)$

(vii) if v is independent of x,y,z, w together, then $\text{coe } (x,y,z,wv) = \text{coe } (x,y,z,w) \text{ ave } v$

again in entire analogy with the properties of the covariance.

20. Explicit formulas in two forms

By using $x-\bar{x}$, $y-\bar{y}$ etc., we have kept the defining equations for the cocumulants moderately short. There will at times be something to be gained by writing these expressions out in detail. They appear as follows:

$$
\begin{aligned}
\mathrm{cok}\ (x,y,z) &= \mathrm{ave}\ xyz - (\mathrm{ave}\ x)(\mathrm{ave}\ yz) - (\mathrm{ave}\ y)(\mathrm{ave}\ xz) \\
&\quad -(\mathrm{ave}\ z)(\mathrm{ave}\ xy) + 2(\mathrm{ave}\ x)(\mathrm{ave}\ y)(\mathrm{ave}\ z) \\
&= \mathrm{ave}\ xyz - (\mathrm{ave}\ x)(\mathrm{ave}\ y)(\mathrm{ave}\ z) \\
&\quad -(\mathrm{ave}\ x)\ \mathrm{cov}\ (y,z) - (\mathrm{ave}\ y)\ \mathrm{cov}\ (x,z) - \\
&\qquad (\mathrm{ave}\ z)\ \mathrm{cov}\ (x,y)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{coe}\ (x,y,z,w) &= \mathrm{ave}\ xyzw - (\mathrm{ave}\ x)(\mathrm{ave}\ yzw) \\
&\quad - (\mathrm{ave}\ y)(\mathrm{ave}\ xzw) - (\mathrm{ave}\ z)(\mathrm{ave}\ xyw) - (\mathrm{ave}\ w)(\mathrm{ave}\ xyz) \\
&\quad - (\mathrm{ave}\ xy)(\mathrm{ave}\ zw) - (\mathrm{ave}\ xz)(\mathrm{ave}\ yw) - (\mathrm{ave}\ xw)(\mathrm{ave}\ yz) \\
&\quad + 2(\mathrm{ave}\ x)(\mathrm{ave}\ y)(\mathrm{ave}\ zw) + 2(\mathrm{ave}\ x)(\mathrm{ave}\ z)(\mathrm{ave}\ yw) \\
&\quad + 2(\mathrm{ave}\ x)(\mathrm{ave}\ w)(\mathrm{ave}\ yz) + 2(\mathrm{ave}\ y)(\mathrm{ave}\ z)(\mathrm{ave}\ xw) \\
&\quad + 2(\mathrm{ave}\ y)(\mathrm{ave}\ w)(\mathrm{ave}\ xz) + 2(\mathrm{ave}\ z)(\mathrm{ave}\ w)(\mathrm{ave}\ xy) \\
&\quad - 6(\mathrm{ave}\ x)(\mathrm{ave}\ y)(\mathrm{ave}\ z)(\mathrm{ave}\ w) \\
&= \mathrm{ave}\ xyzw - (\mathrm{ave}\ x)(\mathrm{ave}\ y)(\mathrm{ave}\ z)(\mathrm{ave}\ w) \\
&\quad - (\mathrm{ave}\ x)\ \mathrm{cok}\ (y,z,w) - (\mathrm{ave}\ y)\ \mathrm{cok}\ (x,z,w) \\
&\quad - (\mathrm{ave}\ z)\ \mathrm{cok}\ (x,y,w) - (\mathrm{ave}\ w)\ \mathrm{cok}\ (x,y,z) \\
&\quad - \mathrm{cov}\ (x,y)\ \mathrm{cov}\ (z,w) - \mathrm{cov}\ (x,z)\ \mathrm{cov}\ (y,w) \\
&\quad - \mathrm{cov}\ (x,w)\ \mathrm{cov}\ (y,z) \\
&\quad - (\mathrm{ave}\ x)(\mathrm{ave}\ y)\ \mathrm{cov}\ (z,w) - (\mathrm{ave}\ x)(\mathrm{ave}\ z)\ \mathrm{cov}\ (y,w) \\
&\quad - (\mathrm{ave}\ x)(\mathrm{ave}\ w)\ \mathrm{cov}\ (y,z) - (\mathrm{ave}\ y)(\mathrm{ave}\ z)\ \mathrm{cov}\ (x,w) \\
&\quad - (\mathrm{ave}\ y)(\mathrm{ave}\ w)\ \mathrm{cov}\ (x,z) -)\mathrm{ave}\ z)(\mathrm{ave}\ w)\ \mathrm{cov}\ (x,y)
\end{aligned}
$$

The apparent lack of symmetry in the basic definitions can be removed by rewriting them in the forms:

$$\text{ave } x = \text{ave } x\,,$$

$$\text{ave } xy = (\text{ave } x)(\text{ave } y) + \text{cov }(x,y),$$

$$\begin{aligned}\text{ave } xyz = {} & (\text{ave } x)(\text{ave } y)(\text{ave } z) \\ & + (\text{ave } x)\text{ cov }(y,z) + (\text{ave } y)\text{ cov }(x,z) \\ & + (\text{ave } z)\text{ cov }(x,y) + \text{cok }(x,y,z),\end{aligned}$$

$$\begin{aligned}\text{ave } xyzw = {} & (\text{ave } x)(\text{ave } y)(\text{ave } z)(\text{ave } w) \\ & + (\text{ave } x)(\text{ave } y)\text{ cov }(z,w) + \text{ave } x)(\text{ave } z)\text{ cov }(y,w) \\ & + (\text{ave } x)(\text{ave } w)\text{ cov }(y,z) + (\text{ave } y)(\text{ave } z)\text{ cov }(x,w) \\ & + (\text{ave } y)(\text{ave } w)\text{ cov }(x,z) + (\text{ave } z)(\text{ave } w)\text{ cov }(x,y) \\ & + \text{cov }(x,y)\text{ cov }(z,w) + \text{cov }(x,z)\text{ cov }(y,w) \\ & \qquad + \text{cov }(x,w)\text{ cov }(y,z) \\ & + \text{ave } x\text{ cok }(y,z,w) + \text{ave } y\text{ cok }(x,z,w) \\ & + \text{ave } z\text{ cok }(x,y,w) + \text{ave } w\text{ cok }(x,y,z) \\ & + \text{coe }(x,y,z,w).\end{aligned}$$

Extension to higher order cumulants is now obvious.

These formulas appear messy, although they are really quite simple. A different notation, which is not as convenient for working with covariances, coskewnesses and coelongations as tools, shows this more clearly. We use it here for immediate expository use only. Put ave x = $\overline{x,}$ cov (x,y) = $\overline{x,y,}$ cok (x,y,z) = $\overline{x,y,z,}$ coe (x,y,z,w) = $\overline{x,y,z,w,}$ then

$$\overline{x,y} = \overline{xy} - \overline{x}\ \overline{y}$$

$$\overline{x,y,z} = \overline{xyz} - \overline{x}\ \overline{yz} - \overline{y}\ \overline{xz} - \overline{z}\ \overline{xy} + 2\overline{x}\ \overline{y}\ \overline{z}$$

$$= \overline{xyz} - \Sigma^{*}\ \overline{a}\ \overline{bc} + 2\overline{x}\ \overline{y}\ \overline{z}$$

$$\overline{x,y,z,w} = \overline{xyzw} - \Sigma^{*}\ \overline{a}\ \overline{bcd} + 2\Sigma^{*}\ \overline{ab}\ \overline{cd} - 6\overline{x}\ \overline{y}\ \overline{z}\ \overline{w}$$

...

$$\overline{xy} = \overline{x}\ \overline{y} + \overline{x,y}$$

$$\overline{xyz} = \overline{x}\ \overline{y}\ \overline{z} + \overline{x}\ \overline{y,z} + \overline{y}\ \overline{x,z} + \overline{z}\ \overline{x,y} + \overline{x,y,z}$$

$$= \overline{x}\ \overline{y}\ \overline{z} + \Sigma^{*}\ \overline{a}\ \overline{b,c} + \overline{x,y,z}$$

$$\overline{xyzw} = \overline{x}\ \overline{y}\ \overline{z}\ \overline{w} + \Sigma^{*}\ \overline{a}\ \overline{b}\ \overline{c,d} + \Sigma^{*}\ \overline{a,b}\ \overline{c,d} + \Sigma^{*}\ \overline{a}\ \overline{b,c,d} + \overline{x,y,z,w}$$

...

where Σ^{*} has the same interpretation as elsewhere in the memorandum, and a,b,c,d is some permutation of x,y,z,w. The formulas are much more perspicuous in this notation. (If the notation were to be extended, var $x = \overset{\square}{x}$, ske $x = \overset{\boxslash}{x}$, elo $x = \overset{\boxtimes}{x}$ might be used.)

21. <u>Independent centered monomials</u>

We next consider the cocumulants of expressions of the form $w_a^{\,i} w_b^{\,j}$ where w_a and w_b are (statistically) independent with moments

$$0,\ \sigma_a^{2},\ \gamma_a\sigma_a^{3},\ \Gamma_a\sigma_a^{4},\ G_a\sigma_a^{5} \text{ and } 0,\ \sigma_b^{2},\ \gamma_b\sigma_b^{3},\ \Gamma_b\sigma_b^{4},\ G_b\sigma^{5}$$

respectively. Using the available relations, these can be easily reduced. Thus, for example,

$$\text{cov}\ (w_a w_b, w_z{}^2 w_b) = \text{ave}\ w_a{}^3 w_b{}^2 - \text{ave}\ w_a w_b\ \text{ave}\ w_a{}^2 w_b$$

$$= \text{ave}\ w_a{}^3\ \text{ave}\ w_b{}^2 - (\text{ave}\ w_a)(.\ .\ .)$$

$$= \gamma_a \sigma_a{}^3 \sigma_b{}^2$$

since ave $w_a = 0$. Similarly, dropping at once terms which clearly vanish,

$$\text{cok}\ (w_a, w_a w_b, w_a{}^2 w_b{}^2) = \text{ave}\ w_a{}^4 w_b{}^3$$

$$- \text{ave}\ w_a{}^2 w_b{}^2\ \text{cov}\ (w_a, w_a w_b)$$

$$= \Gamma_a \sigma_a{}^4 \gamma_b \sigma_b{}^3 - (\sigma_a{}^2 \sigma_b{}^2)(0)$$

$$= \Gamma_a \sigma_a^4 \gamma_b \sigma_b{}^3$$

By similar calculations we reach the results shown in Tables 6 and 7, as well as

$$\text{coe}\ (w_a, w_a, w_a, w_a) = (\Gamma_a - 3)\sigma_a{}^4$$

$$\text{coe}\ (w_a, w_a, w_a, w_b) = 0$$

$$\text{coe}\ (w_a, w_a, w_a, w_a{}^2) = (G_a - 4\gamma_a)\sigma_a{}^5$$

$$\text{coe}\ (w_a, w_a, w_a, w_a w_b) = 0$$

$$\text{coe}\ (w_a, w_a, w_a, w_b{}^2) = 0$$

TABLE 6

Table of coefficients by which $\sigma_a^{i+m}\ \sigma_b^{j+n}$ must be multiplied to obtain the covariance of $w_a^i\ w_b^j$ with $w_a^m\ w_b^n$ when w_a and w_b are (statistically) independent, and each averages zero.

	w_a	w_b	w_a^2	$w_a w_b$	w_b^2	w_a^3
w_a	1	0	γ_a	0	0	Γ_a
w_b	0	1	0	0	γ_b	0
w_a^2	γ_a	0	Γ_a-1	0	0	$G_a-\gamma_a$
$w_a w_b$	0	0	0	1	0	
w_b^2	0	γ_b	0	0	Γ_b-1	
w_a^3	Γ_a	0	$G_a-\gamma_a$	0	0	
$w_a^2 w_b$	0	1	0	γ_a	γ_b	
$w_a w_b^2$	1	0	γ_a	γ_b	0	
w_b^3	0	Γ_b	0	0	$G_b-\gamma_b$	
w_a^4	G_a	0	-	0	0	
$w_a^3 w_b$	0	γ_a	0	Γ_a	$\gamma_a\gamma_b$	
$w_a^2 w_b^2$	γ_a	γ_b	Γ_a-1	$\gamma_a\gamma_b$	Γ_b-1	
$w_a wb^3$	γ_b	0	$\gamma_a\gamma_b$	Γ_b	0	
w_b^4	0	G_b	0	0	-	

Note. With appropriate subscripts var $w = \sigma^2$, ave $w^3 = \gamma\sigma^3$, ave $w^4 = \Gamma\sigma^4$, ave $w^5 = G\sigma^5$.

TABLE 7

Coefficients of $\sigma_a^{i+m+p}\sigma_b^{j+n+q}$ in

$$\text{cok}\,(w_a^i e_b^j,\ w_a^m w_b^n, w_a^p w_b^q)$$

When w_a and w_b are (statistically) independent and each averages zero.

1	2	3	w_a	w_b	w_a^2	$w_a w_b$	w_b^2	w_a^3	$w_a^2 w_b$	$w_a w_b^2$	w_b^3
w_a	w_a		γ_a	0	Γ_a-1	0	0	$G_a-\gamma_a$	0	γ_a	0
w_a	w_b		0	0	0	1	0	0	γ_a	γ_b	0
w_b	w_b		0	γ_b	0	0	Γ_b-1	0	γ_b	0	$G_b-\gamma_b$
w_a	w_a^2		Γ_a-1	0	$G_a-2\gamma_a$	0	0				
w_a	$w_a w_b$		0	1	0	γ_a-1	γ_b				
w_a	w_b^2		0	0	0	γ_b	0				
w_b	w_a^2		0	0	0	γ_a	0				
w_b	$w_a w_b$		1	0	γ_a	γ_b-1	0				
w_b	w_b^2		0	Γ_b-1	0	0	$G_b-2\gamma_b$				

Note. With appropriate subscripts

$$\text{var } w = \sigma^2,$$
$$\text{ave } w^3 = \gamma\sigma^3,$$
$$\text{ave } w^4 = \Gamma\sigma^4,$$
$$\text{ave } w^5 = G\sigma^5.$$

22. Taylor series in independent quantities

Suppose that

$$z = h(w_1, w_2, \ldots, w_k)$$

where ave $w_a = 0$, and the w_a are independent. Then the usual multiple Taylor Series for z can be written in terms of the derivative values at $(0,0,\ldots,0)$, which is the average point, such as

$$h_{aab} = \frac{\partial^3}{\partial {w_a}^2 \partial w_b} h(w_1, w_2, \ldots w_k)\Big|_{(0,0,\ldots,0)}$$

In carrying out this development we will make full use of Σ^*, as used above and discussed in Section 38. When we do, the Taylor series becomes

$$\begin{aligned}
z = {} & h(0,0,\ldots,0) \\
& + \Sigma h_a w_a \\
& + \tfrac{1}{2} \Sigma h_{aa} {w_a}^2 + \Sigma^* h_{ab} w_a w_b \\
& + \tfrac{1}{6} \Sigma h_{aaa} {w_a}^3 + \tfrac{1}{2} \Sigma^* h_{aab} {w_a}^2 w_b + \Sigma^* h_{abc} w_a w_b w_c \\
& + \tfrac{1}{24} \Sigma h_{aaaa} {w_a}^4 + \tfrac{1}{6} \Sigma^* h_{aaab} {w_a}^3 w_b + \tfrac{1}{4} \Sigma^* h_{aabb} {w_a}^2 {w_b}^2 \\
& + \tfrac{1}{2} \Sigma^* h_{aabc} {w_a}^2 w_b w_c + \Sigma^* h_{abcd} w_a w_b w_c w_d \\
& + \ldots
\end{aligned}$$

Where the symmetry of h_{ab} and h_{aabb} in a and b is to be recognized in interpreting Σ^*, and where all the coefficients can be easily checked by direct differentiation--since each monomial appears once and only once.

From this expansion, slightly extended, we immediately deduce

$$\text{ave } z = h(0,0,\ldots,0) + \frac{1}{2}\Sigma h_{aa}\sigma_a^2 + \frac{1}{6}\Sigma h_{aaa}\gamma_a\sigma_a^3$$
$$+\frac{1}{24}\Sigma h_{aaaa}\Gamma_a\sigma_a^4 + \frac{1}{4}\Sigma^* h_{aabb}\sigma_a^2\sigma_b^2 + \frac{1}{120}\Sigma h_{aaaaa}G_a\sigma_a^5$$
$$+\frac{1}{12}\Sigma^* h_{aaabb}\gamma_a\sigma_a^3\sigma_b^2 + \ldots$$

If we think of writing out the variance of z in terms of a double sum of covariances of its terms, and use Table 6, we see that we have

$$\text{var } z = \Sigma h_a^2\sigma_a^2$$
$$+ \Sigma h_a h_{aa}\gamma_a\sigma_a^3$$
$$+\frac{1}{3}\Sigma h_a h_{aaa}\Gamma_a\sigma_a^4 + \frac{1}{4}\Sigma h_{aa}^2(\Gamma_a-1)\sigma_a^4$$
$$+\Sigma^*(h_a h_{abb}+h_{ab}^2+h_{aab}h_b)\sigma_a^2\sigma_b^2$$
$$+\frac{1}{12}\Sigma h_a h_{aaaa}G_a\sigma_a^5 + \frac{1}{6}\Sigma h_{aa}h_{aaa}(G_a-\gamma_a)\sigma_a^5$$
$$+\Sigma^*\left(\frac{1}{2}h_a h_{aabb} + \frac{1}{2}h_{aa}h_{abb} + h_{ab}h_{aab} + \frac{1}{3}h_{aaab}h_b\right)\gamma_a\sigma_a^3\sigma_b^2$$
$$+ \text{terms of order} \geq \sigma^6$$

where the symmetry of the whole coefficient of $\sigma_a^2\sigma_b^2$ is to be recognized in interpreting Σ^* -- a not unmanageable expression.

Writing ske z as a triple sum of coskews of the terms of z, and using Table 7, we find

$$\text{ske } z = \Sigma h_a^{\,3}\gamma_a\sigma^3$$

$$+ \tfrac{3}{2}\,\Sigma h_a^{\,2}h_{zz}(\Gamma_a-1)\,\sigma_a^{\,4} + 6\,\Sigma^* h_a h_b h_{ab}\,\sigma_a^{\,2}\sigma_b^{\,2}$$

$$+ \tfrac{1}{2}\,\Sigma h_a^{\,2}h_{aaa}(G_a-\gamma_a)\sigma_a^{\,5} + \tfrac{3}{4}\,\Sigma h_a h_{aa}^{\,2}(G_a-2\gamma_a)\sigma_a^{\,5}$$

$$+ 3\Sigma^* h_a h_{ab}^{\,2}(\gamma_a-1)\,\sigma_a^{\,3}\sigma_b^{\,2}$$

$$+ \Sigma^*\left(-\tfrac{3}{2}h_a^{\,2}h_{abb} + 3h_{aa}h_{ab}h_b + 3h_a h_{aab}h_b\right)\gamma_a\sigma_a^{\,3}\sigma_b^{\,2}$$

$$+ \text{terms of order} \geq \sigma^6.$$

The same approach to elo z leads to

$$\text{elo } z = \Sigma h_a^{\,4}(\Gamma_a-3)\,\sigma_a^{\,4}$$

$$+ 2\Sigma h_a^{\,3}h_{aa}(G_a-4\gamma_a)\sigma_a^{\,5}$$

$$+ \text{terms of order} \geq \sigma^6.$$

23. <u>The generalized propagation formulas</u>

The formulas in the last section were derived on the , apparently equally important, assumptions that the w_a were independent and had average zero. Suppose them independent, but their averages not to be zero, and put

$$q_a = w_a - \text{ave } w_a ,$$

then the q_a's will satisfy both assumptions, and the formulas will apply in terms of the q_a, provided we expand around $q_a = 0$, i.e. $w_a = \text{ave } w_a$. Thus if

$$\sigma_a^2 = \text{ave } q_a^2 = \text{ave } (w_a - \text{ave } w_a)^2$$

$$\gamma_a \sigma_a^3 = \text{ave } q_a^3 = \text{ave } (w_a - \text{ave } w_a)^3$$

$$\Gamma_a \sigma_a^4 = \text{ave } q_a^4 = \text{ave } (w_a - \text{ave } w_a)^4$$

$$G_a \sigma_a^5 = \text{ave } q_a^5 = \text{ave } (w_a - \text{ave } w_a)^5$$

and if

$$h_{ab} = \frac{\partial^2}{\partial w_a \partial w_b} h(w_1, w_2, \ldots, w_k) \Big|_{(\text{ave } w_1,\ \text{ave } w_2,\ \ldots,\ \text{ave } w_k)}$$

and so on, then the formulas for var z, ske z, and elo z will hold without change. A small change will be needed in the formula for ave z, and it will become

$$\begin{aligned} \text{ave } z = h(&\text{ave } w_1, \text{ave } w_2, \ldots, \text{ave } w_k) \\ &+ \tfrac{1}{2} \Sigma h_{aa} \sigma_a^2 \\ &+ \tfrac{1}{6} \Sigma h_{aaa} \gamma_a \sigma_a^3 \\ &+ \tfrac{1}{24} \Sigma h_{aaaa} \Gamma_a \sigma_a^4 + \tfrac{1}{4} \Sigma^* h_{aabb} \sigma_a^2 \sigma_b^2 \\ &+ \tfrac{1}{120} \Sigma h_{aaaaa} G_a \sigma_a^5 + \tfrac{1}{12} \Sigma^* h_{aaabb} \gamma_a \sigma_a^3 \sigma_b^2 \\ &+ \text{terms of order} \geq \sigma^6. \end{aligned}$$

DETAILS OF FIRST EXAMPLE

24. Derivatives and propagation formulas

We have, writing z for the delay

$$z = h(w_1, \ldots, w_k) = \sqrt{L_1 C_2} + \sqrt{L_3 C_4} + \ldots + \sqrt{L_{2j-1} C_{2j}}$$

$$\frac{\partial h}{\partial L_{2i-1}} = \frac{1}{2}\sqrt{\frac{C_{2i}}{L_{2i-1}}} \quad , \qquad \frac{\partial h}{\partial C_{2i}} = \frac{1}{2}\sqrt{\frac{L_{2i-1}}{C_{2i}}} \quad ,$$

$$\frac{\partial^2 h}{\partial L_{2i-1}\partial C_{2i}} = \frac{1}{4\sqrt{L_{2i-1}C_{2i}}}$$

$$\frac{\partial^2 h}{\partial L_{2i-1}{}^2} = -\frac{1}{4}\sqrt{\frac{C_{2i}}{(L_{2i-1})^3}}$$

and so on.

We wish to illustrate the effect of making, or omitting various transformations. But we would avoid needless complexity. So we shall treat the case where each section has the same design value in considerable detail, and then treat the general case by the quicker methods.

If all sections are to be alike, we may select units of inductance and capacitance such that the design value of each element is 10. This means that $\sigma_a = 1$ for a component with a standard deviation of 10% of its nominal value. Hence values of

σ_a as large as 2 are possible but not unlikely, while values between 1 and 0.1, or possibly slightly smaller may be regarded as likely. This means that we can draw relatively fair impressions from numerically stated formulas involving different powers of σ_a. With this choice, we find

$$h_a = 0.5, \qquad h_{aa} = -\,0.025,$$

$$h_{aaa} = 0.00375, \qquad h_{aaaa} = -\,0.00094^-$$

$$h_{aaaaa} = 0.00033^-,$$

$$h_{ab} = \begin{cases} 0.025, & \text{if } (a,b) = (2i-1,2i) \text{ or } (2i,2i-1) \\ 0, & \text{otherwise} \end{cases}$$

$$h_{aab} = \begin{cases} -0.00125, & \text{if } (a,b) = (2i-1,2i) \text{ or } (2i,\ 2i-1) \\ 0, & \text{otherwise} \end{cases}$$

$$h_{aabb} = \begin{cases} 0.00006^+, & \text{if } (a,b) = (2i-1,2i) \text{ or } (2i,\ 2i-1) \\ 0, & \text{otherwise.} \end{cases}$$

$$h_{aaab} = \begin{cases} 0.00019, & \text{if } (a,b) = (2i-1,2i) \text{ or } (2i,\ 2i-1) \\ 0, & \text{otherwise} \end{cases}$$

$$h_{aaabb} = \begin{cases} -0.00001, & \text{if } (a,b) = (2i-1,\ 2i) \text{ or } (2i,\ 2i-1) \\ 0, & \text{otherwise} \end{cases}$$

hence

$$h_a^2 = 0.25,\ h_a h_{aa} = -0.0125,\ h_a h_{aaa} = 0.00187^+$$

$$h_{aa}^2 = 0.00062^+,\ h_a h_{aaaa} = -0.00047^-,\ h_{aa} h_{aaa} = -.00009^+$$

$$h_a^3 = 0.125,\ h_a^2 h_{aa} = -0.00625,\ h_a^2 h_{aaa} = 0.00094^-,\ h_a h_{aa}^2 = 0.00031^+_,$$

$$h_a^4 = 0.0625,\ h_a^3 h_{aa} = 0.00312^+$$

and, provided $(a,b) = (2i-1,\ 2i)$ or $(2i,\ 2i-1)$ also

$$h_a h_{abb} + h_{ab}^2 + h_{aab} h_b = -\ 0.00062^+$$

$$\frac{1}{2}\ h_a h_{aabb} + \frac{1}{2}\ h_{aa} h_{abb} + h_{ab} h_{aab} + \frac{1}{3}\ h_{aaab} h_b = 0.00003$$

$$6 h_a h_a h_{ab} = 0.0375$$

$$3 h_a h_{ab}^2 = 0.00094^-$$

$$\frac{3}{2}\ h_a^2 h_{abb} + 3 h_{aa} h_{ab} h_b + 3 h_a h_{aab} h_b = -0.00236,$$

these expressions vanishing for all other pairs. We can now write down the propagation formulas, finding

$$\begin{aligned} \text{ave } z = 10j &- 0.0125\ \Sigma\sigma_a^2 \\ &+ 0.00062\ \Sigma\gamma_a\sigma_a^3 - 0.00004\ \Sigma\Gamma_a\sigma_a^4 \\ &+ 0.00002\ \Sigma\sigma_{2i-1}^2\sigma_{2i}^2 \\ &+ 0.00000\ \Sigma\delta_a\sigma_a^5 - 0.00000\ \Sigma(\gamma_{2i-1}\sigma_{2i-1}^3\sigma_{2i}^2 + \gamma_{2i}\sigma_{2i-1}^2\sigma_{2i}^3) \\ &+ \ldots \end{aligned}$$

$$\begin{aligned}
\operatorname{var} z = {} & 0.25\ \Sigma\sigma_a^2 \\
& -0.01250\ \Sigma\gamma_a\sigma_a^3 + 0.00062\ \Sigma\Gamma_a\sigma_a^4 \\
& +0.00016\ \Sigma(\Gamma_a-1)\sigma_a^4 - 0.00062\ \Sigma\sigma_{2i-1}^2\sigma_{2i}^2 \\
& -0.00004\ \Sigma G_a\sigma_a^5 + 0.00002\ \Sigma(G_a-\gamma_a)\sigma_a^5 \\
& +0.00003\ \Sigma(\gamma_{2i-1}\sigma_{2i-1}^3\sigma_{2i}^2 + \gamma_{2i}\sigma_{2i-1}^2\sigma_{2i}^3) \\
& + \ldots
\end{aligned}$$

$$\begin{aligned}
\operatorname{ske} z = {} & 0.125\ \Sigma\ \gamma_a^3 \\
& - 0.00937\ \Sigma(\Gamma_a-1)\sigma_a^4 + 0.0375\ \Sigma\sigma_{2i-1}^2\sigma_{2i}^2 \\
& + 0.00047\ \Sigma(G_a-\gamma_a)\sigma_a^5 + 0.00023\ \Sigma(G_a-2\gamma_a)\sigma_a^5 \\
& + 0.00094\ \Sigma((\gamma_{2i-1}-1)\sigma_{2i-1}^3\sigma_{2i}^2 + (\gamma_{2i}-1)\sigma_{2i-1}^2\sigma_{2i}^3) \\
& - 0.00236\ \Sigma(\gamma_{2i-1}\sigma_{2i-1}^3\sigma_{2i}^2 + \gamma_{2i}\sigma_{2i-1}^2\sigma_{2i}^3 \\
& + \ldots
\end{aligned}$$

$$\operatorname{elo} z = 0.0625\ \Sigma(\Gamma_\alpha-3)\sigma_a^4 + 0.00625\ \Sigma(G_a-4\gamma_a)\sigma_a^5 + \ldots$$

where the index a runs from 1 to $2j$ and the index i runs from 1 to j. We can simplify these somewhat, with the results already quoted in Section 10.

25. Transforming individuals

If now we wish to introduce v's whose partial effects will be linear, we have only to look at

$$h(L_1, C_2, \ldots, L_{2j-1}, C_{2j}) = \sqrt{L_1 C_2} + \ldots + \sqrt{L_{2j-1} C_{2j}}$$

to see that we should take

$$v_{2i-1} = A\sqrt{L_{2i-1}},$$

$$v_{2i} = A\sqrt{C_{2i}},$$

and if we choose A so that the first derivative value is unity, this means

$$v_{2i-1} = 2\sqrt{10}\sqrt{L_{2i-1}},$$

$$v_{2i} = 2\sqrt{10}\sqrt{C_{2i}},$$

$$\sqrt{L_{2i-1}\, C_{2i}} = \frac{1}{40} v_{2i-1}\, v_{2i}$$

whence $z = g(v_1, v_2, \ldots, v_{2j})$

$$= \frac{1}{40}(v_1 v_2 + v_3 v_4 + \ldots + v_{2j-1}\, v_{2j})$$

and

$$g_{2i-1} = \frac{1}{40} v_{2i} = \frac{1}{2}$$

$$g_{2i} = \frac{1}{40} v_{2i-1} = \frac{1}{2} ,$$

$$g_{2i-1,2i} = \frac{1}{40}$$

all other derivative values vanishing. (We note for later use that $t_{2i-1,2i} = 0.1$ with all other t's vanishing.)

The coefficients which appear in the propagation formulas when (a,b) = (2i-1,2i) or (2i, 2i-1) are, omitting those which vanish,

$$g_{ab}{}^2 = \frac{1}{1600} = 0.00062^+ ,$$

$$6g_a g_b g_{ab} = \frac{6}{160} = \frac{3}{80} = 0.0375 ,$$

$$3g_a g_{ab}{}^2 = \frac{3}{3200} = 0.00094^- ,$$

and the resulting propagation formulas are

ave $z = 10j$ + no other terms

var $z = 0.25\ \Sigma\sigma_a^2$

$+\ 0.00062^+\ \Sigma\sigma_{2i-1}^2\sigma_{2i}^2$ + no other terms

$$\text{ske } z = 0.125\ \Sigma\gamma_a\sigma_a^3$$

$$+ 0.0375\ \Sigma\sigma_{2i-1}^2\sigma_{2i}^2$$

$$+ 0.00094\ \Sigma[(\gamma_{2i-1}-1)\sigma_{2i-1}+(\gamma_{2i}-1)\sigma_{2i}]\sigma_{2i-1}^2\sigma_{2i}^2$$

$+$ exactly two sets of terms of order σ^6,

$$\text{elo } z = 0.0625\ \Sigma(\Gamma_a-3)\sigma_a^4$$

$+$ a finite number of terms of orders σ^6 to σ^8.

26. The general case

If we return to the general case of a delay line made of possibly unequal sections, we have (with a change in scale for the v_a for convenience),

$$z = v_1v_2 + v_3v_4 + \ldots + v_{2j-1}v_{2j}$$

with

$$v_{2i-1} = \sqrt{L_{2i-1}},$$

$$V_{2i} = \sqrt{C_{2i}},$$

and

$$t_{2i-1,2i} = \frac{g_{2i-1,2i}}{g_{2i-1}g_{2i}} = \frac{1}{v_{2i}v_{2i-1}} = \frac{1}{t_i}$$

where t_i is the time delay contributed by the ith section, all other t's vanishing. The resulting formulas are

$$\tau_{2i-1}^{2} = (C_{2i}) \operatorname{var} \sqrt{L_{2i-1}} ,$$

$$\tau_{2i}^{2} = (L_{2i-1}) \operatorname{var} \sqrt{C_{2i}} ,$$

and

$$\text{ave } z = \Sigma\, t_i ,$$

$$\text{var } z = \Sigma\, \tau_a^{2} + \Sigma \frac{\tau_{2i-1}^{2}\, \tau_{2i}^{2}}{t_i^{2}} ,$$

$$\text{ske } z = \Sigma\, \gamma_a \tau_a^{3} + \Sigma \frac{6_i \tau_{2i-1}^{2}\, \tau_{2i}^{2}}{t_i} + \Sigma \frac{3[(\gamma_{2i-1}-1)\tau_{2i-1}+(\gamma_{2i}-1)\tau_{2i}]\tau_{2i-1}^{2}\tau_{2i}^{2}}{t_i^{2}}$$

$$+ \text{ terms of order} \geq \tau^{6} ,$$

elo $z = \Sigma(\Gamma_a - 3)\tau_a^{\,4}$

$$+ \text{ terms of order } \geq \tau^6.$$

If we introduce coefficients of variation by

$$\eta_{2i-1} = \frac{\tau_{2i-1}}{t_i} = \frac{\sqrt{C_{2i}}\,\sqrt{\operatorname{var}\sqrt{L_{2i-1}}}}{\sqrt{C_{2i}}\,\sqrt{L_{2i-1}}}$$

$$\eta_{2i} = \frac{\tau_{2i}}{t_i} = \frac{\sqrt{L_{2i-1}}\,\sqrt{\operatorname{var}\sqrt{C_{2i}}}}{\sqrt{L_{2i-1}}\,\sqrt{C_{2i}}}$$

these fall into the form quoted in Section 11.

DETAILS OF SECOND EXAMPLE

27. The response

We next consider a symmetrical π-section attenuating network operating between equal image impedances R. The attenuation can be shown to be given by

$$z = 1 + \frac{1}{2R} R_B + \frac{1}{2} R_B \left\{ \frac{1}{R_A} + \frac{1}{R_C} \right\} + \frac{R}{2} \left\{ \frac{1}{R_A} + \frac{1}{R_C} + \frac{R_B}{R_A R_C} \right\}$$

where R_B is the resistance of the series arm, and R_A and R_C are those of the two shunt arms. The ideal values of R_A, R_B and R_C are

$$R_A = R_C = R \frac{\alpha+1}{\alpha-1} ,$$

$$R_B = R \frac{\alpha^2-1}{2\alpha} ,$$

where α is the design attenuation.

If we place

$$R_A = R \frac{\alpha+1}{\alpha-1} (1 + W_A)$$

$$R_B = R \frac{\alpha^2-1}{2\alpha} (1 + W_B)$$

$$R_C = R \frac{\alpha+1}{\alpha-1} (1 + W_C)$$

the attenuation becomes

$$z = \alpha (1+W_A)^{-1} (1+W_C)^{-1} \left[1 + W_B \frac{\alpha-1}{\alpha+1} + (W_A+W_C) \frac{\alpha+3}{2(\alpha+1)} \right.$$

$$\left. + W_B (W_A+W_C) \frac{\alpha-1}{2\alpha} + W_A W_C \frac{\alpha^2+4\alpha-1}{4\alpha^2} + W_A W_C W_B \frac{\alpha^2-1}{4\alpha^2} \right]$$

and we are almost ready to find derivatives.

Accurate differentiation will however be swifter and easier if we rewrite this expression as a sum of products of functions of the single W's, namely as:

$$z = \alpha \frac{1}{1+W_A} \frac{1}{1+W_C} + \alpha \frac{\alpha-1}{\alpha+1} \frac{1}{1+W_A} W_B \frac{1}{1+W_C} + \alpha \frac{\alpha+3}{2(\alpha+1)} \frac{W_A}{1+W_A} \frac{1}{1+W_C}$$

$$+ \alpha \frac{\alpha+3}{2(\alpha+1)} \frac{1}{1+W_A} \frac{W_C}{1+W_C} + \frac{\alpha-1}{2} \frac{W_A}{1+W_A} W_B \frac{1}{1+W_C}$$

$$+ \frac{\alpha-1}{2} \frac{1}{1+W_A} W_B \frac{W_C}{1+W_C} + \frac{\alpha^2+4\alpha-1}{4\alpha} \frac{W_A}{1+W_A} \frac{W_C}{1+W_C}$$

$$+ \frac{\alpha^2-1}{4\alpha} \frac{W_A}{1+W_A} W_B \frac{W_C}{1+W_C}$$

and note that the values of the successive derivatives, at $W = 0$, of $(1+W)^{-1}$ and $W/(1+W)$ are as follows:

order of diff'n:	0	1	2	3	4
$(1+W)^{-1}$	1	-1	2	-6	24
$W/(1+W)$	0	1	-2	6	-24

We thus obtain, writing

$$z = H(W_A, W_B, W_C) \quad ,$$

and introducing

$$\beta = \frac{\alpha}{2}\frac{\alpha-1}{\alpha+1} \;, \; \gamma = \frac{(\alpha-1)^3}{4\alpha(\alpha+1)} \;, \; \delta = \frac{(\alpha-1)^2}{2(\alpha+1)}$$

which satisfy $2\beta\gamma = \delta^2$,

$H_A = H_C = -\beta$, $H_{AA} = H_{CC} = 2\beta$, $H_{AAA} = H_{CCC} = -6\beta$

$H_{AAAA} = H_{CCCC} = 24\beta$, $H_{AAAAA} = H_{CCCCC} = -120\beta$

$H_{AC} = \gamma$, $H_{AAC} = H_{ACC} = -2\gamma$, $H_{AACC} = 4\gamma$, $H_{AAAC} = H_{ACCC} = 6\gamma$,

$H_{AAACC} = -12\gamma$,

$H_B = 2\beta$

$H_{AB} = H_{BC} = -\delta$, $H_{AAB} = H_{BCC} = 2\delta$, $H_{AAAB} = H_{BCCC} = -6\delta$,

$H_{BB} = H_{ABB} = H_{AABB} = H_{BBC} = H_{BBCC} = H_{ABBC} = 0$

We are now almost ready to write down the generalized propagation formulas.

It is convenient to first evaluate certain coefficients, with the results given in Table 8.

TABLE 8

Values of Coefficients in the Second Example

Coefficient	Value When ab=AB or CB	ab=BA or BC	ab=AC or CA
$H_a H_{abb} + H_{ab}^2 + H_{aab} H_b$	$\delta^2+4\beta\delta$	$\delta^2+4\beta\delta$	$\gamma^2+4\beta\gamma$
$\frac{1}{2} H_a H_{aabb} + \frac{1}{2} H_{aa} H_{abb}$ $+ H_{ab} H_{aab} + \frac{1}{3} H_{aaab} H_b$	$-2\delta^2-4\beta\delta$	0	$-6\beta\gamma-2\gamma^2$
$6 H_a H_b H_{ab}$	$12\beta^2\delta$	$12\beta^2\delta$	$6\beta^2\gamma$
$3 H_a H_{ab}^2$	$-3\beta\delta^2$	$6\beta\delta^2$	$-3\beta\gamma^2$
$\frac{3}{2} H_a^2 H_{abb} + 3 H_{aa} H_{ab} H_b + 3 H_a H_{aab} H_b$	$-24\beta^2\delta$	$12\beta^2\delta$	$-15\beta^2\gamma$

Coefficient	Value When a=B	a=A or C
H_a^2	$4\beta^2$	β^2
$H_a H_{aa} = \frac{1}{12} H_a H_{aaaa}$	0	$-2\beta^2$
$\frac{1}{3} H_a H_{aaa} = -\frac{1}{6} H_{aa} H_{aaa}$	0	$2\beta^2$
$\frac{1}{4} H_{aa}^2$	0	β^2
H_a^3	$8\beta^3$	$-\beta^3$
$\frac{3}{2} H_a^2 H_{aa}$	0	$3\beta^3$
$\frac{1}{2} H_a^2 H_{aaa} = \frac{3}{4} H_a H_{aa}^2$	0	$-3\beta^3$

Using these results, the formulas become

$$
\begin{aligned}
\text{ave } z = \alpha & \\
& + \beta\,(\sigma_A{}^2 + \sigma_C{}^2) \\
& - \beta\,(\gamma_A\sigma_A{}^3 + \gamma_B\sigma_C{}^3) \\
& + \beta\,(\Gamma_A\sigma_A{}^4 + \Gamma_C\sigma_C{}^4) \\
& + \gamma\sigma_A{}^2\sigma_C{}^2 \\
& - \beta\,(G_A\sigma_A{}^5 + G_C\sigma_C{}^5) \\
& + \gamma\,(\gamma_A\sigma_A + \gamma_C\sigma_C)\sigma_A{}^2\sigma_C{}^2 \\
& + \ldots
\end{aligned}
$$

$$
\begin{aligned}
\text{var } z = {} & \beta^2\,(\sigma_A{}^2 + 4\sigma_B{}^2 + \sigma_C{}^2) \\
& - 2\beta^2\,(\gamma_A\sigma_A{}^3 + \gamma_C\sigma_C{}^3) \\
& + 2\beta^2\,(\Gamma_A\sigma_A{}^4 + \Gamma_C\sigma_C{}^4) \\
& + \beta^2\left[(\Gamma_A-1)\sigma_A{}^4 + (\Gamma_C-1)\sigma_C{}^4\right] \\
& + (\delta^2 + 4\beta\delta)\sigma_B{}^2\,(\sigma_A{}^2 + \sigma_C{}^2) + (\gamma^2 + 4\beta\gamma)\sigma_A{}^2\sigma_C{}^2 \\
& - 2\beta^2\,(G_A\sigma_A{}^5 + G_C\sigma_C{}^5) \\
& - 2\beta^2\left[(G_A-\gamma_A)\sigma_A{}^5 + (G_C-\gamma_C)\sigma_C{}^5\right] \\
& - (2\delta^2 + 4\beta\delta)(\gamma_A\sigma_A{}^3 + \gamma_C\sigma_C{}^3)\sigma_B{}^2 - (2\gamma^2 - 6\beta\gamma)(\gamma_A\sigma_A + \gamma_C\sigma_C)\sigma_A{}^2\sigma_C{}^2 \\
& + \ldots
\end{aligned}
$$

$$
\begin{aligned}
\text{ske } z = & -\beta^3 (\gamma_A\sigma_A{}^3 - 8\gamma_B\sigma_B{}^3 + \gamma_C\sigma_C{}^3) \\
& + 3\beta^3 \left[(\Gamma_A-1)\sigma_A{}^4 + (\Gamma_C-1)\sigma_C{}^4\right] \\
& + 12\beta^2\delta(\sigma_A{}^2 + \sigma_C{}^2)\sigma_B{}^2 + 6\beta^2\gamma\sigma_A{}^2\sigma_C{}^2 \\
& - 3\beta^3 \left[(G_A-\gamma_A)\sigma_A{}^5 + (G_C-\gamma_C)\sigma_C{}^5\right] \\
& - 3\beta^3 \left[(G_A-2\gamma_A)\sigma_A{}^5 + (G_C-2\gamma_C)\sigma_C{}^5\right] \\
& - 3\beta\delta^2 \left[(\gamma_A-1)\sigma_A{}^3\sigma_B{}^2 - 2(\gamma_B-1)\sigma_B{}^3(\sigma_A{}^2+\sigma_C{}^2) + (\gamma_C-1)\sigma_C{}^3\sigma_B{}^2\right] \\
& - 3\beta\gamma^2 \left[(\gamma_A-1)\sigma_A + (\gamma_C-1)\sigma_C\right]\sigma_A{}^2\sigma_C{}^2 \\
& - 6\beta^2\delta \left[(4\gamma_A\sigma_A{}^3 + 4\gamma_C\sigma_C{}^3)\sigma_B{}^2 + 2\gamma_B\sigma_B{}^3(\sigma_A{}^2+\sigma_C{}^2)\right] \\
& - 13\beta^2\gamma(\gamma_A\sigma_A + \gamma_C\sigma_C)\sigma_A{}^2\sigma_C{}^2 \\
& + \cdots
\end{aligned}
$$

$$
\begin{aligned}
\text{elo } z = & \beta^4 \left[(\Gamma_A-3)\sigma_A{}^4 + 16\,(\Gamma_B-3)\sigma_B{}^4 + (\Gamma_C-3)\sigma_C{}^4\right] \\
& - 4\beta^4 \left[(G_A-4\gamma_A)\sigma_A{}^5 + (G_C-4\gamma_C)\sigma_C{}^5\right] \\
& + \cdots
\end{aligned}
$$

To bring these results into a more perspicuous form, we change to a more natural scale by putting

$$W_A = 0.1\, w_A,\ W_B = 0.1\, w_B,\ W_C = 0.1\, w_C$$

$$z = H(W_A, W_B, W_C) = h(w_A, w_B, w_C)$$

so that the standard deviations of the w's will be of the order of unity or somewhat smaller. Notice that β, γ and δ are substantial fractions of α, for example

α:	1	2	5	10	∞
β:	0	0.17α	0.33α	0.41α	0.50α
γ:	0	0.02α	0.11α	0.17α	0.25α
δ:	0	0.08α	0.27α	0.37α	0.50α

so that we may, roughly, think of α, β, γ and δ as of about the same order. The resulting formulas were already given in Section 12.

28. Transforming Individuals.

If we revert to the original expression for the transmission

$$z = 1 + \frac{1}{2R} R_B + \frac{1}{2} R_B \left\{\frac{1}{R_A} + \frac{1}{R_C}\right\} + \frac{R}{2} \left\{\frac{1}{R_A} + \frac{1}{R_C} + \frac{R_B}{R_A R_C}\right\}$$

and consider the effect of varying R_A, R_B or R_C alone, we see that z is linear in $1/R_C$. Thus the appropriate transforms for R_A and R_C arise by replacing resistances by conductances, while the appropriate transform for R_B is the identity.

If we place

$$\frac{1}{R_A} = \frac{1}{R} \frac{\alpha-1}{\alpha+1} (1 + V_A)$$

$$R_B = R \frac{\alpha^2-1}{2\alpha} (1 + V_B)$$

$$\frac{1}{R_C} = \frac{1}{R} \frac{\alpha-1}{\alpha+1} (1 + V_C)$$

then the attenuation becomes

$$z = \alpha + \frac{\alpha}{2} \frac{\alpha-1}{\alpha+1} (V_A+2V_B+V_C) + \frac{(\alpha-1)^3}{4\alpha(\alpha+1)} V_A V_C + \frac{(\alpha-1)^2}{2(\alpha+1)} V_B(V_A+V_C)$$

$$+ \frac{(\alpha-1)^3}{4\alpha(\alpha+1)} V_A V_B V_C$$

or, in abbreviated notation

$$z = \alpha + \beta(V_A+2V_B+V_C) + \gamma V_A V_C + \delta V_B(V_A+V_C) + \gamma V_A V_B V_C$$

and if we write $z = f(V_A, V_B, V_C)$ then

we have

$$f_A = f_C = \beta, \; f_{AC} = \gamma$$

$$f_B = 2\beta, \; f_{AB} = f_{BC} = \delta,$$

$$f_{ABC} = \gamma$$

while all other derivatives vanish for $V_A = V_B = V_C = 0$. Thus the generalized propagation formulas reduce to :

$$\begin{aligned}
\text{ave } z &= \alpha \\
&\quad + \ldots \\
\text{var } z &= \beta^2(\sigma_A{}^2 + 4\sigma_B{}^2 + \sigma_C{}^2) \\
&\quad + \delta^2(\sigma_A{}^2 + \sigma_C{}^2)\sigma_B{}^2 + \gamma^2\sigma_A{}^2\sigma_C{}^2 \\
&\quad + \ldots \\
\text{ske } z &= \beta^3(\gamma_A\sigma_A{}^3 + 8\gamma_B\sigma_B{}^3 + \gamma_C\sigma_C{}^3) \\
&\quad + 12\beta^2\delta(\sigma_A{}^2 + \sigma_C{}^2)\sigma_B{}^2 + 6\beta^2\gamma\sigma_A{}^2\sigma_C{}^2 \\
&\quad + 3\beta\delta^2\left[(\gamma_A-1)\sigma_A{}^3 + (\gamma_C-1)\sigma_C{}^3)\right]\sigma_B{}^2 \\
&\quad + 3\beta\gamma^2\left[(\gamma_A-1)\sigma_A + (\gamma_C-1)\sigma_C\right]\sigma_A{}^2\sigma_C{}^2 \\
&\quad + 6\beta\delta^2(\gamma_B-1)\sigma_B{}^3(\sigma_A{}^2 + \sigma_C{}^2) \\
&\quad + \ldots \\
\text{elo } z &= \beta^4\left[(\Gamma_A-3)\sigma_A{}^4 + 16(\Gamma_B-3)\sigma_B{}^4 + (\Gamma_C-3)\sigma_C{}^4\right] \\
&\quad + \ldots
\end{aligned}$$

With the possible expection of the formula for ske z, where the reduction in number of terms is, though still large in number only about half of the number present, the gain over each of the previous formulas is notable.

Rescaling again by writing

$$V_A = 0.1\ v_A$$

$$V_B = 0.1\ v_B$$

$$V_C = 0.1\ v_C$$

we obtain the formulas already given in Section 12.

DETAILS OF TRANSFORMATION OF RESPONSE

29. Strategy

As indicated in Part II, the attack by transformation can involve both transformation of the individual [component] variables, which is naturally directed to annulling the higher unmixed derivative values, and transformation of the response [system] variable, which might naturally be directed at either increasing the normality of the response distribution or at reducing the size of some of the terms involving mixed derivative values. It appears that transformation of the individual variables is not only easier but rather more efficacious. Thus we will be wise to plan to transform the individual variables first. This will be true, even although an incautious transformation of the response would disturb the desirable situation obtained by transforming individuals, since we can be cautious, and arrange for a compensating transformation of the individual variables to accompany the transformation of the response.

We thus contemplate the following sequence of situations:

$$\begin{cases} z = h(w_1, w_2, \ldots, w_k) \\ \text{perfectly general} \end{cases}$$

$$\begin{cases} z = g(v_1, v_2, \ldots, v_k) \\ v_a \text{ a function of } w_a \\ g_{aa} = g_{aaa} = \ldots = 0 \end{cases}$$

$$\begin{cases} y = f(u_1, u_2, \ldots, u_k \\ y = \varphi(z) \\ v_a = v_a(u_a) \\ f_{aa} = f_{aaa} = \ldots = 0 \end{cases}$$

Here we may as well require that the first derivative values of ea h v_a with respect to w_a, of y with respect to z, and of each v_a with respect to u_a all be unity in order to keep all leading terms the same. We naturally denote the latter two derivative values, evaluated at the corresponding average points, by φ' and v_a'. We may do this because we are dealing with functions of single variables. We shall use φ'', φ''', φ^{iv}, v_a'', v_a''', v_a^{iv} for the corresponding higher derivatives.

We have already discussed the first step in some detail (Section 6ff.) so that we need here to be concerned with the second step. If we are to carry out the second step in practice, we need to be able to find out about $v_a(u_a)$ and about f_{ab}'s in terms of φ (presumably in terms of the derivative values of φ) and the g_{ab}. This we shall do next.

30. Transfer formula details

Suppose now that, starting with

$$z = g(v_1, v_2, \ldots, v_k)$$

where $g_{aa} = g_{aaa} = g_{aaaa} = \ldots = 0$, we transform the response by $y = \varphi(z)$, and wish to make transformations on the individual variables $v_a = v_a(u_a)$ so that in

$$y = \varphi(g(v_1(u_1), v_2(u_2), \ldots v_k(u_k))) = f(u_1, u_2, \ldots u_k)$$

we will have $f_{aa} = f_{aaa} = f_{aaaa} = \ldots = 0$. We surely need to have formulas expressing the (successive and) cross derivative values of f in terms of the successive and cross derivative values of g and the derivative values of φ. Information on the functions $v_a(u_a)$ will also be helpful.

There is no loss of generality in assuming that the first derivative values φ', v_1', v_2', ..., v_k' are all unity. With this convention, the higher simple successive derivatives of f are:

$$f_a = \varphi' g_a v_a' = g_a$$

$$f_{aa} = \varphi''(g_a v_a')^2 + \varphi' g_{aa}(v_a')^2 + \varphi' g_a v_a'' = \varphi'' g_a^2 + g_{aa} + g_a v_a''$$

$$\begin{aligned} f_{aaa} &= \varphi'''(g_a v_a')^3 + 3\varphi'' g_{aa} g_a (v_a')^3 + 3\varphi'' g_a^2 v_a' v_a'' \\ &\quad + \varphi' g_{aaa}(v_a')^3 + 3\varphi' g_{aa} v_a'' v_a' + \varphi' g_a v_a''' \\ &= \varphi''' g_a^3 + 3\varphi'' g_{aa} g_a + 3\varphi'' g_a^2 v_a'' + g_{aaa} \\ &\quad + 3\, g_{aa} v_a'' + g_a v_a''' \end{aligned}$$

$$f_{aaaa} = \varphi^{iv}(g_a v_a')^4 + 6\varphi''' g_{aa} g_a^2 (v_a')^4 + 6\varphi''' g_a^3 v_a''(v_a')^2$$

$$+ 4\varphi'' g_{aaa} g_a (v_a')^4 + 18\varphi'' g_{aa} g_a (v_a')^2 v_a''$$

$$+ 3\varphi'' g_a^2 (v_a'')^2 + 4\varphi'' g_a^2 v_a' v_a''' + \varphi' g_{aaaa} (v_a')^4$$

$$+ 6\varphi' g_{aaa} (v_a')^2 v_a'' + 3\varphi' g_{aa} (v_a'')^2 + 4\varphi' g_{aa} v_a' v_a'''$$

$$+ \varphi' g_a v_a^{iv} + 3\varphi'' g_{aa}^2 (v_a')^4$$

$$= \varphi^{iv} g_a^4 + 6\varphi''' g_{aa} g_a^2 + 6\varphi''' g_a^3 v_a'' + 4\varphi'' g_{aaa} g_a$$

$$+ 18\varphi'' g_{aa} g_a v_a'' + 3\varphi'' g_a^2 (v_a'')^2 + 4\varphi'' g_a^2 v_a'''$$

$$+ g_{aaaa} + 5\, g_{aaa} v_a'' + 3\, g_{aa} (v_a'')^2 + 4\, g_{aa} v_a'''$$

$$+ g_a v_a^{iv} + 3\varphi'' g_{aa}^2$$

If now we set these successively equal to zero, starting with the second and using $g_{aa} = g_{aaa} = g_{aaaa} = \ldots = 0$, we obtain

$$0 = \varphi'' g_a^2 + g_a v_a''$$

$$0 = g''' g_a^3 + 3\varphi'' g_a^2 v_a'' + g_a v_a'''$$

$$0 = \varphi^{iv} g_a^4 + 6\varphi''' g_a^3 v_a'' + 3\varphi'' g_a^2 (v_a'')^2 + 4\varphi'' g_a^2 v_a''' + g_a v_a^{iv}$$

and similarly

$$0 = \varphi^{v} g_a^5 + 10\varphi^{iv} g_a^4 v_a'' + 10\varphi''' g_a^3 v_a''' + 5\varphi'' g_a^2 v_a^{iv} + g_a v_a^{v}$$

$$+ 15\varphi''' g_a^3 (v_a'')^2 + 10\varphi'' g_a^2 v_a''' v_a''$$

whence

$$v_a'' = -(g_a \varphi''),$$

$$v_a''' = -g_a^2 \varphi''' - 3\varphi'' g_a v_a''$$

$$= -(g_a^2 \varphi''') + 3(g_a \varphi'')^2$$

$$= g_a^2(-\varphi''' + 3\varphi''\varphi''),$$

$$v_a^{iv} = -g_a^3 \varphi^{iv} - 6\, g_a^2 \varphi^{iii} v_a'' - 3\varphi'' g_a (v_a'')^2 - 4\varphi'' g_a v_a'''$$

$$= -(g_a^3 \varphi^{iv}) + 10(g_a^2 \varphi''')(g_a \varphi'') - 15(g_a \varphi'')^3$$

$$= g_a^3(-\varphi^{iv} + 10\varphi'''\varphi'' - 15\varphi''\varphi''\varphi'').$$

and

$$v_a^{v} = g_a^4[-\varphi^{v} + 15\varphi^{iv}\varphi'' + 10\varphi'''\varphi''' - 105\varphi'''(\varphi'')^2 + 105(\varphi'')^4].$$

with these formulas in hand, we are prepared for the cross deriv- atives, and find

$$f_{ab} = \varphi'' g_a v_a' g_b v_b' + \varphi' g_{ab} v_a' v_b'$$

$$= \varphi'' g_a g_b ,$$

$$f_{abb} = \varphi''' g_a v_a'(g_b v_b')^2 + 2\varphi'' g_b g_{ab} v_a'(v_b')^2 + \varphi'' g_a v_a' g_{bb}(v_b')^2$$

$$+ \varphi'' g_a v_a' g_b v_b'' + \varphi' g_{abb} v_a'(v_b')^2$$

$$+ \varphi' g_{ab} v_a' v_b''$$

$$= \varphi''' g_a g_b^2 + 2\varphi'' g_{ab} g_b + \varphi'' g_a g_{bb} + \varphi'' g_a g_b v_b'' + g_{abb} + g_{ab} v_b''$$

$$= g_{abb} + \varphi'' g_{ab} g_b + \varphi'' g_a g_{bb} + g_a g_b^2(\varphi''' - \varphi''\varphi'')$$

$$f_{abbb} = \varphi^{iv}(g_a v_a')(g_b v_b')^3 + 3\varphi''' g_a v_a' g_b g_{bb}(v_b')^3$$

$$+ 3\varphi''' g_a v_a' g_b^2 v_b' v_b'' +$$

$$+ \varphi'' g_a v_a' g_{bbb}(v_b')^3 + 3\varphi'' g_a v_a' g_{bb}(v_b')(v_b'')$$

$$+ 3\varphi''' g_{ab} g_b^2 v_a'(v_b')^3 + 3\varphi'' g_{abb} g_b v_a'(v_b')^3$$

$$+ 3\varphi'' g_{ab} g_{bb} v_a'(v_b')^3 + 6\varphi'' g_{ab} g_b v_a' v_b' v_b''$$

$$+ \varphi'' g_a v_a' g_b v_b'''$$

(formula continues)

$$+ \varphi' g_{abbb}(v_a')(v_b')^3$$

$$+ 3\varphi' g_{abb} v_a' v_b' v_b'' +$$

$$+ \varphi' g_{ab} v_a' v_b'''$$

$$= \varphi^{iv} g_a g_b^3 + 3\varphi''' g_a g_b g_{bb} + 3\varphi''' g_a g_b^2 v_b'' + 3\varphi''' g_{ab} g_b^2$$

$$+ \varphi'' g_a g_b v_b''' + \varphi'' g_a g_{bbb} + 3\varphi'' g_a g_{bb} v_b'' + 3\varphi'' g_{abb} g_b$$

$$+ 3\varphi'' g_{ab} g_{bb}$$

$$+ 6\varphi'' g_{ab} g_b v_b'' + g_{abbb} + 3g_{abb} v_b'' + g_{ab} v_b'''$$

$$= g_{abbb} + 3\varphi'' g_{ab} g_{bb} + \varphi'' g_a g_{bbb} + (2\varphi''' - 3\varphi''\varphi'') g_{ab} g_b^2$$

$$+ 3(\varphi''' - \varphi''\varphi'') g_a g_b g_{bb} + (\varphi^{iv} - 4\varphi''\varphi''' + 3\varphi''\varphi''\varphi'') g_a g_b^3$$

$$= g_{abbb} + (2\varphi''' - 3\varphi''\varphi'') g_{ab} g_b^2 + (\varphi^{iv} - 4\varphi''\varphi''' + 3\varphi''\varphi''\varphi'') g_a g_b^3$$

$$f_{aabb} = \varphi^{iv}(g_a v_a')^2 (g_b v_b')^2 + \varphi''' g_{aa}(v_a')^2 (g_b v_b')^2$$

$$+ \varphi''' g_a v_a (g_b v_b')^2 + 4\varphi''' g_a g_b g_{ab}(v_a')^2 (v_b')^2$$

$$+ 2\varphi'' g_{ab} g_{ab}(v_a')^2 (v_b')^2 + 2\varphi'' g_{aab} g_b (v_a')^2 (v_b')^2$$

$$+ 2\varphi'' g_{ab} g_b v_a''(v_b')^2$$

$$+ \varphi''' g_a^2 (v_a')^2 g_{bb}(v_b')^2 + \varphi'' g_{aa}(v_a')^2 g_{bb}(v_b')^2$$

$$+ 2\varphi'' g_a (v_a')^2 g_{abb}(v_b')^2$$

$$+ \varphi'' g_a v_a'' g_{bb}(v_b')^2 + \varphi''' g_a^2 (v_a')^2 g_b v_b'' + \varphi'' g_{aa}(v_a')^2 g_b v_b''$$

$$+ 2\varphi'' g_a (v_a')^2 g_{ab} v_b'' + \varphi'' g_a v_a'' g_b v_b''$$

$$+ \varphi' g_{aabb}(v_a')^2 + \varphi' g_{abb} v_a''(v_b')^2$$

$$+ \varphi' g_{aab}(v_a')^2 v_b'' + \varphi' g_{ab} v_a'' v_b''$$

$$= \varphi^{iv} g_a^2 g_b^2 + \varphi''' g_{aa} g_b^2 + \varphi''' g_a g_b^2 v_a'' + 4\varphi''' g_a g_b g_{ab}$$

$$+ \varphi''' g_a^2 g_{bb} + \varphi''' g_a^2 g_b v_b'' + 2\varphi'' g_{ab}^2 + 2\varphi'' g_{aab} g_b$$

(formula continues)

$$+ 2\varphi'' g_{ab} g_b v_a'' + \varphi'' g_{aa} g_{bb} + 2\varphi'' g_a g_{abb} + \varphi'' g_a g_{bb} v_a''$$

$$+ \varphi'' g_{aa} g_b v_b'' + 2\varphi'' g_a g_{ab} v_b'' + \varphi'' g_a g_b v_a'' v_b'' + g_{aabb}$$

$$+ g_{abb} v_a'' + g_{aab} v_b'' + g_{ab} v_a'' v_b''$$

$$= g_{aabb} + \varphi'' g_{aab} g_b + \varphi'' g_a g_{abb} + \varphi'' g_{aa} g_{bb}$$

$$+ 2\varphi'' g_{ab}^2 + (\varphi''' - \varphi''\varphi'') g_a^2 g_{bb} + (\varphi''' - \varphi''\varphi'') g_{aa} g_b^2$$

$$+ (4\varphi''' - 3\varphi''\varphi'') g_a g_{ab} g_b + (\varphi^{iv} - 2\varphi'''\varphi'' + \varphi''\varphi''\varphi'') g_a^2 g_b^2$$

$$= g_{aabb} + \varphi''(g_{aab} g_b + g_a g_{abb}) + 2\varphi'' g_{ab}^2$$

$$+ (4\varphi''' - 3\varphi''\varphi'') g_a g_{ab} g_b + (\varphi^{iv} - 2\varphi'''\varphi'' + \varphi''\varphi''\varphi'') g_a^2 g_b^2$$

(In each case, the first equals sign is followed by the general expression, the second by the result of using unity for φ', v_a' and v_b', the third by the result of further substituting for v_a'', v_b'', and v_b''', and the last by the result of using $g_{bb} = g_{bbb} = 0$.)

These results are to be supplemented, when needed, by

$$v_a = \text{ave } v_a + u_a + \frac{1}{2} g_a(-\varphi'') u_a^2$$

$$+ \frac{1}{6} g_a^2(-\varphi''' + 3\varphi''\varphi'') u_a^3$$

$$+ \frac{1}{24} g_a^3(-\varphi^{iv} + 10\varphi''' \varphi'' - 15\varphi''\varphi''\varphi'') u_a^4$$

$$+ \cdots$$

which inverts into the Taylor expansion

$$u_a = (v_a - \text{ave } v_a)$$

$$+ \frac{1}{2} g_a \varphi''(v_a - \text{ave } v_a)^2$$

$$+ \frac{1}{6} g_a^2 \varphi'''(v_a - \text{ave } v_a)^3$$

$$+ \frac{1}{24} g_a^3 \varphi^{iv}(v_a - \text{ave } v_a)^4$$

$$+ \ldots$$

as it obviously should, since

$$u_a = \frac{1}{g_a} [\varphi(g(\text{ave } v_1, \text{ave } v_2, \ldots, v_a, \ldots, \text{ave } v_k))$$

$$-\varphi(g(\text{ave } v_1, \text{ave } v_2, \ldots, \text{ave } v_a, \ldots \text{ave } v_k))].$$

When the formulas giving f's in terms of g's are translated into formulas giving s's in terms of t's, we obtain the formulas already given in Section 14.

31. Improving normality

We are now ready to consider the possible modes of use of transformations of response. First, consider the possibility of choosing φ so that the normality of $g = \varphi(z)$ will be improved.

The leading terms in ske y, where $y = f(u_1, u_2, \ldots, u_k)$ are

$$\Sigma\, \gamma_a \tau_a^3 + \Sigma^*(6s_{ab})\tau_a^2\tau_b^2$$

$$= \Sigma\, \gamma_a \tau_a^3 + \Sigma^*(6t_{ab} + 6\varphi'')\tau_a^2\tau_b^2$$

where the τ_a are calculated for the u_a. (Hopefully, but not certainly, the next term will not be important.)

In order to reduce the size of this approximation to the skewness, we may proceed along two different paths. We may try to make the effects of some terms compensate others, or we may try to make as many individual terms small as we can.

The difficulty in seeking reduction by compensation stems from our lack of knowledge of the σ_a and γ_a. Setting the expression equal to zero, and solving for φ'' yields

$$\varphi'' = -\frac{\Sigma\gamma_a\tau_a^3 + 6\Sigma^*\, t_{ab}\tau_a^2\tau_b^2}{6\Sigma^*\tau_a^2\,\tau_b^2}$$

which is rather unmanageable if the τ_a^2 and γ_a are either not precisely known or may vary. In some circumstances such reduction might be practical, but its general utility is no better than doubtful.

The difficulty in seeking reduction by reducing individual terms is two-fold. There are terms which remain unaffected. And not all affected terms vanish for the same value of φ''. The straightforward condition for eliminating a particular term is, however,

$$\varphi'' \approx - t_{ab} \, .$$

Clearly, most attention should be given to large values of $\tau_a \tau_b$. If we weight by $\tau_a^2 \, \tau_b^2$, we find

$$\varphi'' = - \frac{\Sigma \, t_{ab} \, \tau_a^2 \, \tau_b^2}{\tau_a^2 \, \tau_b^2}$$

We may not know the τ's well enough to use them at all accurately, but we may get a rough idea of a possibly useful φ'' from this relation.

Further arguments against the usefulness of attempts to improve normality can be based on the relative size of the γ_a and the τ_a. If the former are much the larger, then φ'' will be large, and other terms in ske y may well be important. If the τ_a are large, the same may be said. If all are small, and the γ_a are smaller, then the skewness is small — why do we concern ourselves with it?

32. <u>Simplifying formulas by eliminating terms</u>

Clearly we must choose which formula is to be simplified in which way. (Obviously we are going to try to simplify the variance formula — but it may be worthwhile to rationalize this choice.) The arguments of the last section speak strongly against simplifying the formulas for skewness or elongation. The second term in the formula for the average involves values of 4^{th} derivatives $[f_{aabb}]$. So our attention is directed to the variance, where the terms next after the first involve $\tau_a^2\tau_b^2$. If we look for only the lowest order of differentiation we are led to

$$\Sigma\, s_{ab}^2\, \tau_a^2\tau_b^2$$

and matters proceed as in the second approach to increased normality, since larger values of $\tau_a^2\tau_b^2$ deserve greater weight. If, again, we can disregard variations in the sizes of the σ_a, weighting by $g_a^2g_b^2$, which leads again to

$$\varphi'' = -\frac{\Sigma\, g_a g_b g_{ab}}{\Sigma\, g_a^2\, g_b^2}$$

may be of use. All this concerns only part of the coefficient of $\tau_a^2\tau_b^2$, however!

If we concern ourselves with the whole coefficient of $\tau_a^2\tau_b^2$, we face

$$s_{aab} + s_{ab}^2 + s_{abb}$$

which is

$$(t_{aab} + t_{ab}^2 + t_{abb}) + 4t_{ab}\varphi'' + (2\varphi''' - \varphi''\varphi'')$$

By choosing φ''' and φ'' appropriately we can make this vanish for any two pairs (a,b) or for any two sets of equivalent pairs. As the examples will show, this may sometimes be useful.

33. Power transformations

The natural transformations to consider are often of the form

$$y = A(z + c)^p$$

whence

$$\varphi' = pA(z_o+c)^{p-1},$$

$$\varphi'' = p(p-1)\ A(z_o+c)^{p-2},$$

$$\varphi''' = p(p-1)(p-2)A(z_o+c)^{p-3},$$

and when we fix φ' at 1, these yield

$$\varphi'' = (p-1)/(z_o+c)$$

$$\varphi''' = (p-1)(p-2)/(z_o+c)^2$$

$$2\varphi''' - \varphi''\varphi'' = [2(p-1)(p-2) - (p-1)^2]/(z_o+c)^2$$

$$= (p-1)(p-3)/(z_o+c)^2$$

$$= \frac{p-3}{p-1}\,(\varphi'')^2$$

whence p may be found from

$$1 - p = \frac{\varphi''\varphi''}{\varphi''' - \varphi''\varphi''}$$

$z_o + c$ as

$$z_o + c = \frac{1-p}{-\varphi''}$$

and A as

$$A = \frac{(z_o+c)^{1-p}}{p} \quad .$$

It is worthy of remark that the case $p = 0$ corresponds to

$$y = (z_o+c) \log(z+c)$$

with $\varphi' = 1$, $\varphi'' = -1/(z_o+c)$, $\varphi''' = 2/(z_o+c)^2$, and $\varphi^{iv} = -6/(z_o+c)^3$, as might have been expected.

Similarly, as $p \to \pm \infty$, the transform approaches

$$\frac{1}{\left(\frac{p}{z_o+c}\right)} e^{\left(\frac{p}{z_o+c}\right)(z-z_o)}$$

where

$$\varphi' = 1,\ \varphi'' = \left(\frac{p}{z_o+c}\right),$$

$$\varphi''' = \left(\frac{p}{z_o+c}\right)^2,\ \varphi^{iv} = \left(\frac{p}{z_o+c}\right)^3$$

Especially since the values of p, c and A may be quite sensitive to the values of φ'' and φ''', it is useful to have formulas for φ^{iv} and φ^{v} in terms of φ'' and φ'. Both for the general and limiting cases of power transformations, we have

$$\varphi^{iv} = (2\varphi''' - \varphi''\varphi'')\,\frac{\varphi'''}{\varphi''}$$

and

$$\varphi^{v} = (3\varphi''' - 2\varphi''\varphi'')\,\frac{\varphi^{iv}}{\varphi''}$$

so that the more complicated formulas connecting s's and t's become

$$s_{abbb} = t_{abbb} + (2\varphi'''-3\varphi''\varphi'')t_{ab} + (2\varphi'''-\varphi''\varphi'')\frac{\varphi'''}{\varphi''} - \varphi''(4\varphi'''-3\varphi''\varphi'')$$

$$s_{aabb} = t_{aabb} + \varphi''(t_{aab}+2t_{ab}^2+t_{abb}) + (4\varphi'''-3\varphi''\varphi'')t_{ab}$$

$$+ (2\varphi'''-\varphi''\varphi'')\left(\frac{\varphi'''}{\varphi''} - \varphi''\right)$$

$$s_{aaabb} = t_{aaabb} + 6\varphi''\, t_{ab}\, t_{aab} + 6(\varphi'''-\varphi''\varphi'')t_{aab}$$

$$+ (2\varphi'''-3\varphi''\varphi'')t_{abb} + 6(\varphi'''-\varphi''\varphi'')t_{ab}^2$$

$$+ \frac{1}{\varphi''}\left\{12(\varphi''')^2 - 22\varphi'''\varphi''\varphi'' + 9(\varphi''\varphi'')^2\right\} t_{ab}$$

$$+ \frac{1}{(\varphi'')^2}\left\{6(\varphi''')^3 - 16(\varphi''')^2\,\varphi''\varphi'' + 13\varphi'''(\varphi''\varphi'')^2-3(\varphi''\varphi'')^3\right\}$$

We are really more concerned with formulas for the coefficients in the propagation formulas. Combination of the formulas just given, and the use of suitable abbreviations, lead to the formulas already given in Table 5 (Section 18).

If we seek to fix φ'' and φ''' by arranging for the individual coefficients of $\tau_a^2\tau_b^2$ and $\tau_c^2\tau_d^2$ to both vanish, then, solving the simultaneous linear equations, we must take

$$\varphi'' = \frac{1}{4}\,\frac{q_{ab} - q_{cd}}{t_{ab} - t_{cd}} \;,$$

$$2\varphi''' - \varphi''\varphi'' = \frac{t_{cd}q_{ab} - t_{ab}\,q_{cd}}{t_{ab} - t_{cd}} \;,$$

where

$$q_{ab} = t_{aab} + t_{ab}^2 + t_{abb} \;,$$

$$q_{cd} = t_{ccd} + t_{cd}^2 + t_{cdd} \;.$$

TRANSFORMATION OF RESPONSE IN THE EXAMPLES

34. Details for first example

After we had made the individual transformations, we had

$$g_{2i-1} = g_{2i} = \frac{1}{2}$$

$$g_{2i-1,2i} = \frac{1}{40}$$

with all other higher derivative values zero. Hence $t_{2i-1,2i} = 0.1$, while all other t's vanish.

If we try to reduce individual coefficients in the skewness, we have, when we equate φ'' to $-t_{ab}$ for various choices of (a,b)

$$\varphi'' \approx -0.1 \text{ (j occurrences)}$$

$$\varphi'' \approx 0.0 \text{ (2j(j-1) occurrences)}$$

and clearly we are likely to prefer to keep $\varphi'' = 0$ unless $j = 1$. The weighted solution gives

$$\varphi'' \approx -\frac{1}{10(2j-1)}$$

which is clearly quite near zero.

For the most plausible approach we find

$$0 = 0.01 + 0.4\varphi'' + 2\varphi''' - \varphi''\varphi'' \qquad (j \text{ times})$$

$$0 = 0 + 0 + 2\varphi''' - \varphi''\varphi'' \qquad (2j(j-1) \text{ times})$$

and we can satisfy both if

$$\varphi'' = -1/40$$

$$\varphi''' = 1/3200$$

which corresponds to a power transformation with $p = 3$, $z_0 + c = -80$, $A = 1/(3(-80)^2)$, namely to

$$y = \frac{(z-10j-80)^3}{19{,}200}$$

for which $\varphi' = 1, \varphi'' = -1/40$, $\varphi''' = 1/3200$, and $\varphi^{iv} = \varphi^{v} = \ldots = 0$.

The resulting values of s_{ab}, etc. are, when a and b are paired (that is when $(a,b) = (2i-1, 2i)$ or $(2i, 2i-1)$),

$$s_{ab} = 0.1 - 0.025 = 0.075$$

$$s_{aab} = s_{abb} = -0.025(0.1) + \frac{1}{3200} - \frac{1}{1600} = -0.00281^{+}$$

$$s_{abbb} = s_{aaab} = \left(\frac{2}{3200} - \frac{3}{1600}\right)(0.1) + \frac{4}{(40)3200}$$

$$+ \frac{3}{(-40)^3} = -0.00014^{-}$$

$$s_{aabb} = -0.025(2)(0.1)^2 + \left(\frac{4}{3200} - \frac{3}{1600}\right)(0.01) - \frac{2}{(40)3200}$$

$$+ \frac{1}{(-40)^3}$$

$$= -0.00056^{+}$$

$$s_{aaabb} = s_{aabbb} = 6\left(\frac{1}{3200} - \frac{1}{1600}\right)(0.1)^2 + \left(\frac{16}{(40)(3200)}\right.$$

$$\left. + \frac{9}{(40)^3}\right)(0.1) - \frac{1}{(3200)^2} + \frac{7}{(3200)(40)^2} - \frac{3}{(40)^4}$$

$$= +0.00009$$

These lead to the following coefficients (still to 5 decimals).

$$\frac{1}{4} s_{aabb} = -0.00014^{+},\ \frac{1}{12} s_{aaabb} = +0.00001^{+}$$

$$s_{aab} + s_{ab}^{2} + s_{abb} = 0$$

$$\frac{1}{2} s_{aabb} + s_{ab}s_{aab} + \frac{1}{3} s_{aaab} = -0.00054$$

$$6s_{ab} = 0.45 \qquad 3s_{ab}^{2} = .01688^{-}$$

$$3s_{aab} + \frac{3}{2} s_{abb} = -0.01265^{-}$$

and hence to the formulas

$$\text{ave } y = -80$$

$$-0.00014\ \Sigma\ \tau_{2i-1}\tau_{2i}^{2}$$

$$+0.00001\ \Sigma(\gamma_{2i-1}\tau_{2i-1}+\gamma_{2i})\tau_{2i-1}^{2}\tau_{2i}^{2}$$

$$+ \text{ terms of order } \geq\tau^{6},$$

$$\text{var } y = \Sigma\tau_{a}^{2}$$

$$-0.00052\ \Sigma(\gamma_{2i-1}\tau_{2i-1} + \gamma_{2i}\tau_{2i})\tau_{2i-1}^{2}\tau_{2i}^{2}$$

$$+ \text{ terms of order } \geq\tau^{6},$$

$$\text{ske } y = \Sigma\gamma_a\tau_a^{3}$$

$$+ 0.45\ \Sigma\ \tau_{2i-1}^{2}\tau_{2i}^{2}$$

$$+ 0.01688^{-}\ \Sigma[((\gamma_{2i-1}-1)\tau_{2i-1}+(\gamma_{2i}-1)\tau_{2i}]\tau_{2i-1}^{2}\tau_{2i}^{2}]$$

$$- 0.01265^{-}\ \Sigma(\gamma_{2i-1}\tau_{2i-1} + \gamma_{2i}\tau_{2i})\tau_{2i-1}^{2}\tau_{2i}^{2}$$

$$+ \text{terms of order} \geq \tau^{6}\ ,$$

$$\text{elo } y = \Sigma(\Gamma_a-3)\tau_a^{4}$$

$$+ \text{terms of order} \geq \tau^{6}.$$

These formulas are to be compared with the corresponding formulas before transformation, which are

$$\text{ave } z = 10j + \text{no other terms},$$

$$\text{var } z = \Sigma\tau_a^{2}$$

$$+ 0.01\ \Sigma\tau_{2i-1}^{2}\tau_{2i}^{2}$$

$$+ \text{no other terms},$$

$$\text{ske } z = \Sigma\gamma_a\tau_a^{\,3}$$

$$+ 0.6\ \Sigma\tau_{2i-1}^{\,2}\tau_{2i}^{\,2}$$

$$+ 0.03\ \Sigma[(\gamma_{2i-1}-1)\tau_{2i-1} + (\gamma_{2i}-1)\tau_{2i}]\tau_{2i-1}^{\,2}\tau_{2i}^{\,2}$$

$$+ \text{terms of order} \geq \tau^6 ,$$

$$\text{elo } z = \Sigma(\Gamma_a-3)\tau_a^{\,4}$$

$$+ \text{terms of order} \geq \tau^6 .$$

These two sets of formulas are to be assessed in view of the fact that $\tau_a = \frac{1}{2}\sigma_a$, so that τ's as large as 1.0 are possible but unlikely. Values between 0.5 and 0.05 are more plausible. Thus

$$- 0.0005\ (\gamma_{2i-1}\tau_{2i-1} + \gamma_{2i}\tau_{2i})\tau_{2i-1}^{\,2}\tau_{2i}^{\,2}$$

is not likely to be more than a twentieth the magnitude of $+0.01\tau_{2i-1}^{\,2}\tau_{2i}^{\,2}$ and might be considerably less.

In general terms, the transformation has

(i) added some minor, almost surely negligible, terms to the formula for the average,

(ii) reduced the most important correction term in the variance from perhaps 1% of the leading term to perhaps 0.05% -- from almost certain negligibility to utter unimportance.

(iii) modified the correction terms in the skewness, and has had some slight tendency to reduction of coefficients.

Overall, transformation has done what it was asked to do -- it has eliminated the most important correction term in the formula for the variance. However, in this instance, the correction term in the variance was not important enough to warrant much effort.

35. A salutary example

In searching for an example showing how much transformation of the response could matter if we had picked the wrong terms of response initially, we may well ask what would have happened it we had started with a response proportional to the cube of the delay time of the filter. We can easily calculate the reduced derivative values by taking

$$y = \frac{z^3}{300j^2}$$

when $\varphi' = 1$, $\varphi'' = 1/5j$, $\varphi''' = 1/50j^2$, $\varphi^{iv} = \varphi^{v} = \ldots = 0$.

The s_{ab} etc., which now do not refer to well chosen terms of response, become

$$s_{ab} = 0.1 + 0.2/j$$

$$s_{aab} = 0.02/j + 0.02/j^2 - 0.04/j^2 = 0.02/j - 0.02/j^2$$

$$s_{abbb} = (0.04/j^2 - 0.12/j^2)0.1 - 4(0.004/j^3 + 3(0.008/j^3)$$

$$= - 0.008/j^2 + 0.008/j^3$$

$$s_{aabb} = (0.2/j)(0.02) + (4(0.02/j^2) - 3(0.04/j^2))\ (0.1)$$

$$- 2(0.004/j^3) + (0.008/j^3)$$

$$= 0.004/j - 0.004/j^2$$

The leading and first correction terms in the variance become

$$\text{var } y = \Sigma\tau_a^{\ 2}$$

$$+ (0.01 + 0.08/j)\ \Sigma\ \tau_{2i-1}^{\ 2}\ \tau_{2i}^{\ 2}$$

$$+ \ldots$$

For very small j (namely $j = 1$), beginning with the cube of the delay time as the response <u>might</u> make the first correction term relevant, but for larger j (namely $j \geq 2$) the first correction is small enough to be unimportant, even when this very poor choice of terms is made.

36. Details for second example

After we had made the individual transformations, we had

$$t_{AC} = \frac{\gamma}{\beta^2} = \frac{\alpha^2-1}{\alpha^3} = \frac{1}{\alpha} - \frac{1}{\alpha^3}$$

$$t_{AB} = t_{BC} = \frac{\delta}{2\beta^2} = \frac{\alpha+1}{\alpha^2} = \frac{1}{\alpha} + \frac{1}{\alpha^2}$$

$$t_{ABC} = \frac{\gamma}{2\beta^3} = \frac{(\alpha+1)^2}{\alpha^4} \approx \frac{1}{\alpha^2}$$

with all other t's vanishing.

If we try to reduce individual coefficients in the skewness, we have

$$-\varphi'' \approx \frac{1}{\alpha} - \frac{1}{\alpha^3} \qquad \text{(once)}$$

$$-\varphi'' \approx \frac{1}{\alpha} + \frac{1}{\alpha^2} \qquad \text{(twice)}$$

and the choice $\varphi'' = -1/\alpha$ seems likely to be a helpful compromise. The choice $p = 0$, $c = 0$, since $z_o = \alpha$, will do this. Thus

$$y = \alpha \log z$$

,

is to be considere . The weighted solution differs from $\varphi'' = -1/\alpha$ by a small, presumably negligible result. If we try to delete mixed correction terms in the variance, we find

$$\left(\frac{1}{\alpha} + \frac{1}{\alpha^2}\right)^2 + 4\left(\frac{1}{\alpha} + \frac{1}{\alpha^2}\right)\varphi'' + 2\varphi''' - \varphi''\varphi'' = 0 \quad \text{(twice)}$$

$$\left(\frac{1}{\alpha} - \frac{1}{\alpha^3}\right)^2 + 4\left(\frac{1}{\alpha} - \frac{1}{\alpha^3}\right)\varphi'' + 2\varphi''' - \varphi''\varphi'' = 0 \quad \text{(once)}$$

which leads to

$$\varphi'' = -\frac{1}{2\alpha} - \frac{1}{4\alpha^2} + \frac{1}{4\alpha^3}$$

$$\varphi''' = \frac{5}{8\alpha^2} - \frac{11}{32\alpha^4} \quad \frac{3}{32\alpha^4} + \frac{7}{16\alpha^5} + \frac{1}{32\alpha^6}$$

We might reasonably hope to obtain good results with $\varphi'' = -1/2\alpha$ and $\varphi''' = 5/8\alpha^2$ which correspond to a power transformation with $p = 1/3$, $z_o + 3 = 4\alpha/3$, $c = \alpha/3$, and $A = 3(4\alpha/3)^{2/3}$. Hence

$$y = 3^{1/3}(4\alpha)^{2/3}\,(z + \alpha/3)^{1/3}$$

with $\varphi' = 1$, $\varphi'' - \frac{1}{2\alpha}$, $\varphi''' = 5/8\alpha^2$, $\varphi^{iv} = -5/4\alpha^3$ is indicated.

If we write $t_{ab} = \frac{1}{\alpha} + \Delta$, then we have from Table 5 and the vanishing of all t's with 3 or more subscripts which appear therein

$$D = \frac{5}{8\alpha^2} - \frac{1}{4\alpha^2} = \frac{3}{8\alpha^2}\ ,\ E = \frac{1}{4\alpha^2}$$

$$s_{aabb} = \frac{2}{2\alpha}\left(\frac{1}{\alpha} + \Delta\right)^2 + \left(\frac{12}{8\alpha^2} + \frac{1}{4\alpha^2}\right)\left(\frac{1}{\alpha} + \Delta\right)$$

$$- 2\alpha \frac{3}{8\alpha^2}\left(\frac{6}{8\alpha^2} + \frac{1}{4\alpha^2}\right)$$

$$= - \frac{1}{4\alpha^2}\Delta - \frac{1}{\alpha}\Delta^2$$

$$s_{aaabb} = \left(-\frac{18}{8\alpha^2}\right)\left(\frac{1}{\alpha} + \Delta\right)^2 - 2\alpha\left(\frac{108}{64\alpha^4} + \frac{6}{32\alpha^4} - \frac{1}{16\alpha^4}\right)$$

$$\left(\frac{1}{\alpha} + \Delta\right) + 4\alpha^2\left(\frac{162}{512\alpha^6} + \frac{18}{256\alpha^6} - \frac{3}{128\alpha^6}\right)$$

$$= - \frac{2}{\alpha^4} - \frac{9}{2\alpha^3}\Delta - \frac{9}{4\alpha^2}\Delta^2$$

$$s_{aab} + s_{ab} + s_{abb} = \left(\frac{1}{\alpha} + \Delta\right)^2 - \frac{2}{\alpha}\left(\frac{1}{\alpha} + \Delta\right) + \frac{1}{\alpha^2} = \Delta^2$$

$$\left.\begin{array}{l}\frac{1}{2} s_{aabb} + s_{ab}\, s_{aab} \\ \quad + \frac{1}{3} s_{aaab}\end{array}\right\} = - \frac{1}{\alpha}\left(\frac{1}{\alpha} + \Delta\right)^2 + \left(\frac{11}{8\alpha^2} + \frac{7}{24\alpha^2}\right)\left(\frac{1}{\alpha} + \Delta\right)$$

$$- 2\alpha\left(\frac{15}{64\alpha^4} + \frac{7}{64\alpha^4}\right)$$

$$= - \frac{1}{48\alpha^3} - \frac{1}{3\alpha^2}\Delta - \frac{1}{\alpha}\Delta^2$$

$$s_{ab} = (\frac{1}{\alpha} + \Delta) - \frac{1}{2\alpha} = \frac{1}{2\alpha} + \Delta$$

$$s_{ab}{}^2 = \frac{1}{4\alpha^2} + \frac{1}{\alpha}\Delta + \Delta^2$$

$$3s_{aab} + \frac{3}{2} s_{abb} = - \frac{9}{4\alpha}(\frac{1}{\alpha} + \Delta) + \frac{27}{16\alpha^2} = - \frac{9}{16\alpha^2} - \frac{9}{4\alpha}\Delta$$

When we put $\Delta = 1/\alpha^2$ (for AB or BC) and $- 1/\alpha^3$ (for AC) we obtain

ave $y = y_o$

$$- \left\{\frac{1}{16\alpha^4} + \frac{1}{4\alpha^5}\right\}\left\{\tau_A{}^2\tau_B{}^2 + \tau_B{}^2\tau_C{}^2\right\} + \left\{\frac{1}{16\alpha^5} - \frac{1}{4\alpha^7}\right\}\tau_A{}^2\tau_C{}^2$$

$$- \left\{\frac{8}{48\alpha^4} - \frac{3}{8\alpha^5} - \frac{3}{16\alpha^6}\right\}(\gamma_A\tau_A{}^3\tau_B{}^2 + \gamma_B\tau_B{}^3(\tau_A{}^2 + \tau_C{}^2)$$

$$+ \gamma_C\tau_C{}^3\tau_B{}^2)$$

$$- \left\{\frac{8}{49\alpha^4} + \frac{3}{8\alpha^6} + \frac{-3}{16\alpha^8}\right.(\gamma_A\tau_A + \gamma_C\tau_C)\tau_A{}^2\tau_C{}^2$$

+ terms of order $\geq \tau^6$

$$\text{var } y = \tau_A{}^2 + \tau_B{}^2 + \tau_C{}^2$$

$$+ \frac{1}{\alpha^4}(\tau_A{}^2\tau_B{}^2 + \tau_B{}^2\tau_C{}^2) + \frac{1}{\alpha^6}\tau_A{}^2\tau_C{}^2$$

$$- \left\{\frac{7}{4\alpha^3} + \frac{1}{\alpha^4} + \frac{3}{2\alpha^5}\right\}(\gamma_A\tau_A{}^3\tau_B{}^2 + \gamma_B\tau_B{}^3(\tau_A{}^2 + \tau_C)^2 + \gamma_C\tau_C{}^3\tau_B{}^2)$$

$$- \left\{\frac{7}{4\alpha^3} - \frac{1}{\alpha^5} + \frac{3}{2\alpha^7}\right\}(\gamma_A\tau_A + \gamma_C\tau_C)\tau_A{}^2\tau_C{}^2$$

+ terms of order $\geq \tau^6$

$$\text{ske } y = \gamma_A\tau_A{}^3 + \gamma_B\tau_B{}^3 + \gamma_C\tau_C{}^3$$

$$+ \left\{\frac{3}{\alpha} + \frac{6}{\alpha^2}\right\}\tau_B{}^2(\tau_A{}^2 + \tau_C{}^2) + \left\{\frac{3}{\alpha} - \frac{6}{\alpha^3}\right\}\tau_A{}^2\tau_C{}^2$$

$$+ \left\{\frac{3}{4\alpha^2} + \frac{3}{\alpha^3} + \frac{3}{\alpha^4}\right\}\Big[(\gamma_A-1)\tau_A{}^3\tau_B{}^2+(\gamma_B-1)\tau_B{}^3(\tau_A{}^2+\tau_C{}^2)$$

$$+(\gamma_C-1)\tau_A{}^3\tau_B{}^2\Big]$$

$$+ \left\{\frac{3}{4\alpha^2} - \frac{3}{\alpha^4} + \frac{3}{\alpha^6}\right\}\Big[(\gamma_A-1)\tau_A{}^3\tau_C{}^2+(\gamma_C-1)\tau_C{}^3\tau_A{}^2\Big]$$

$$+ \left\{-\frac{9}{16\alpha^2} - \frac{9}{4\alpha^3}\right\}(\gamma_A\tau_A{}^3\tau_B{}^2+\gamma_B\tau_B{}^3(\tau_A{}^2+\tau_C{}^2)+\gamma_C\tau_C{}^3\tau_B{}^2)$$

$$+ \left\{-\frac{9}{16\alpha^2} + \frac{9}{4\alpha^4}\right\}(\gamma_A\tau_A{}^3\tau_C{}^2+\gamma_C\tau_A{}^2\tau_C{}^3)$$

$$+ \text{ terms of order } \geq \tau^6$$

$$\text{elo } y = \Sigma(\Gamma_a-3)\tau_a{}^4$$

$$+ \text{ terms of order } \geq \tau^6.$$

If we put $\alpha = 10$, then the variance formula becomes, to five decimals,

$$\text{var } y = \tau_A^{\,2} + \tau_B^{\,2} + \tau_C^{\,2}$$

$$+ 0.00010\ \tau_B^{\,2}(\tau_A^{\,2} + \tau_C^{\,2})$$

$$- 0.00002\ (\gamma_A\tau_A^{\,3}\tau_B^{\,2} + \gamma_B\tau_B^{\,3}(\tau_A^{\,2} + \tau_C^{\,2}) + \gamma_C\tau_C^{\,3}\tau_A^{\,2})$$

$$- 0.00002\ (\gamma_A\tau_A^{\,3}\tau_C^{\,2} + \gamma_C\tau_A^{\,2}\tau_C^{\,3})$$

$$+ \ldots$$

and clearly the correction terms are all very negligible. Before transformation, we had

$$t_{AB} = t_{BC} = \frac{\delta}{2\beta^2} = \frac{1}{\alpha} + \frac{1}{\alpha^2} = 0.11,$$

$$t_{AC} = \frac{\gamma}{\beta^2} = \frac{1}{\alpha} - \frac{1}{\alpha^3} = 0.099\ ,$$

and the variance formula was

$$\text{var } z \approx \tau_A^{\,2} + \tau_B^{\,2} + \tau_C^{\,2}$$

$$+ 0.01210\ \tau_B^{\,2}\ (\tau_A^{\,2} + \tau_C)^2$$

$$+ 0.00980\ \tau_A^{\,2}\tau_C^{\,2}$$

Thus we have reduced the first correction term by a factor of 100 or so, although it was already rather small.

37. More accurate analysis

The above results were based on the plausible-seeming assumption that the leading terms in φ'' and φ''' suffice to select an adequate transformation. It will prove enlightening to inquire into what the "best" transformation proves to be for $\alpha = 10$. We find, using all terms, $\varphi'' = -.05225, \varphi''' = .004\ 870\ 03125$

$$1 - p = \frac{.002\ 730\ 062\ 5}{.002\ 139\ 968} = 1.2757^{+}$$

$$z_o + c = 24.416$$

$$A = \frac{24.416^{1.2757}}{-.275749} = -213.705$$

$$y = -213.705\ (z+14.416)^{-.2757}$$

This transformation is clearly quite different in appearance from the previous one, although its derivative values are quite similar, namely

	Approximate	Exact
φ^{ii}	-.0500	-.05225
φ^{iii}	.00625	.00487
φ^{iv}	-.00125	-.00065
φ^{v}	.00034	.00011

X GLOSSARIES AND NOTATION

38. <u>The Σ^* notation</u>.

We have used at various points the very convenient, but not standard, Σ^* notation. With its convenience there comes a need for some care in its use. At this point, after recalling its definition, some examples will be provided. In general, Σ^* implies the summation of all the different terms of the form written after it which can be obtained without identifying subscripts. In this, its usage corresponds to our usual practice in writing down simple formulas. We write

$$(x_1+x_2)^2 = x_1^2+x_2^2+2x_1x_2$$

but

$$(x_1+x_2)(y_1+y_2) = x_1y_1+x_2y_2+x_1y_2+x_2y_1$$

Similarly we now write

$$(\Sigma x_a)^2 = \Sigma x_a^2+2\Sigma^* x_a x_b$$

but

$$(\Sigma x_a)(\Sigma y_b) = \Sigma x_a y_a+\Sigma^* x_a y_b.$$

Note the $2\Sigma^*$ in one formula and the Σ^* without the two in the other. This exactly corresponds to our writing $2x_1x_2$ in $(x_1+x_2)^2$ and $1(x_1y_2+x_2y_1)$ in $(x_1\ x_2)(y_1+ y_2)$.

As a consequence we must be careful of what happens when we specialize variables, break up terms, or multiply expressions. Thus if we put $y_b=x_b$ in $\Sigma^* x_a y_b$ the answer is $2\Sigma^* x_a x_b$ and not merely $\Sigma^* x_a x_b$. As another example, consider the following

$$\begin{aligned}\Sigma^* x_a y_b &= \Sigma^*(x_a y_b - x_a y_a + x_a y_a) \\ &= \Sigma^*(x_a(y_b - y_a) + x_a y_a) \\ &= \Sigma^* x_a(y_b - y_a) + (n-1)\ \Sigma x_a y_a\end{aligned}$$

where we find n-1 times $\Sigma x_a y_a = \Sigma^* x_a y_a$ rather than merely $\Sigma^* x_a y_a$. (Here n = the number of values taken on by a.) Finally, consider $(\Sigma^* x_a{}^2 x_b)^2$, which we now write

$$\begin{aligned}(\Sigma^* x_a{}^2 x_b)(\Sigma^* x_c{}^2 x_d) = {} & \Sigma^* x_a{}^4 x_b{}^2 \\ & +2\ \Sigma^* x_a{}^3 x_b{}^3 \\ & +2\ \Sigma^* x_a{}^4 x_b x_d \\ & +2\ \Sigma^* x_a{}^2 x_b{}^3 x_d \\ & +6\ \Sigma^* x_a{}^2 x_b{}^2 x_c{}^2 \\ & +\ \Sigma^* x_a{}^2 x_b x_c{}^2 x_d\ .\end{aligned}$$

It would not be too difficult to miss some of these numerical coefficients.

All these needs for care considered, however, the advantages of the Σ^* notation seem to outweigh the disadvantages.

39. Glossary of statistical and other special terms used.

average Arithmetic mean, especially of a probability distribution. (within range of standard usage.)

average point The situation in which each individual variable takes on its average value. (special)

<u>cocumulant</u> A seminvariant measure of mutual variation. Definitions of the first three and statement of their properties given in Section 19. (new, not yet standard)

<u>coelongation</u> The fourth cocumulant -- a function of four arguments. (See Section 19 for definition and properties. (new, not yet standard)

<u>correlated</u> Not having zero covariance. (standard)

<u>coskewness</u> The third cocumulant -- a function of three arguments. See Section 19 for definition and properties. (new, not yet standard)

<u>covariance</u> The average product of deviations from means -- the second cocumulant. See Section 19 for definition and properties. (standard)

<u>cumulant (or seminvariant)</u> A seminvariant polynomial in the moments of a distribution; a coefficient in the expansion of the logarithm of the moment generating function in terms of $f^k/k!$. The first cumulants are average, variance, skewness and elongation. (standard)

<u>cumulative normal distribution</u> Function expressing a normal distribution in terms of the total probability of values less than any given value. (standard)

<u>deviate</u> A quantity with a probability distribution whose average is zero. (standard)

<u>derivative value</u> A value of a derivative at a preassigned point. (not yet widely used)

<u>elongation</u> The fourth cumulant, usually a measure of longtailedness. See also section 4. (new)

<u>independence (statistical)</u> Two chance quantities are statistically independent when knowledge of the value of one gives no information about the probability distribution of the other. (standard)

<u>moment</u> An average value of some power of the quantity concerned. See also Sections 2 to 4. (standard)

<u>normal distribution</u> A particularly simple shape of distribution of probability which can be characterized in many ways ; as the distribution of the sum of an indefinitely large number of independent quantities, as a distribution all of whose higher cumulants vanish, as an example of the normal "bell-shaped curve". Also "Gaussian" or "Maxwellion". (standard)

<u>partial effect</u> The effect of varying one individual variable, while holding all the others constant. (not yet standard)

<u>relative derivative values</u> Ratios of higher derivative values to the corresponding products of first derivative values. (not yet standard)

<u>relative moments</u> Ratios of higher moments about the mean to that power of the standard deviation which makes the result dimensionless. (not infrequent)

<u>response</u> A variable whose value is regarded as caused by the values of another variable (stimulus) or other variables (individual variables, stimuli). (standard)

<u>seminvariant</u> (see cumulant)

<u>skewness</u> The third cumulant (= the third of moment about the mean), usually a measure of asymmetry in long-tailedness. See also Section 4. (marginal)

<u>standard measure</u> A distribution is expressed in standard measure when its average is zero and its variance unity. (standard)

<u>standardized deviate</u> A quantity with average zero and unit variance. (standard)

<u>terms</u> The type of unit in which a variable is expressed to be contrasted with "scale" which indicates the size of the unit used. (The variable "temperature" for example, can be expressed in terms of degrees, log degrees, reciprocal degrees, etc.) (new--no clear pattern of usage)

<u>variance</u> The root-mean-square deviation from the average; the second cumulant. See also Section 2. (standard)

<u>40. Glossary of abbreviations.</u>

"ave" Average value of expression following. (Used by some authors.)

"coe(-,-,-,-)" coelongation of four expressions appearing as arguments (not yet standard) "cok(-,-,-)" coskewness of three expressions appearing as arguments. (not yet standard)

"cov(-,-)" covariance of two expressions appearing as arguments (standard)

"elo" elongation (fourth cumulant) of expression following. (not yet standard)

"ske" skewness (third cumulant) of expression following. (not yet standard)

"var" variance (second cumulant) of expression following. (standard)

41. <u>Notation used here for response functions.</u>

The main topic of this memorandum deals with the behavior of a response as a function of individual variables. The following notations are used consistently

(a) No special assumptions:

$$z = h(w_1, w_2, \ldots, w_k)$$

(b) Higher unmixed derivatives all vanish:

$$z = g(v_1, v_2, \ldots, v_k)$$

(c) Transformed response, with individual variables such that all higher unmixed derivatives all vanish:

$$y = f(u_1, u_2, \ldots, u_k)$$

In addition, various other notations are used transiently, including

$$y = f(x_1, x_2, \ldots, x_k),$$

$$z = g^{(1)}(v_1^{(1)}, v_2^{(1)}, \ldots, v_k^{(1)}),$$

$$z = H(W_A, W_B, W_C),$$

$$z = h(w_A, w_B, w_C),$$

$$z = f(V_A, V_B, V_C),$$

with restrictions stated in context.

42. Index of notations used "on the line"

The index to symbols which follows is divided into three sections: first, in this section, symbols used "on the line", next, in the next section, symobls used as subscripts, and finally, in section 44, symbols used as exponents on other superscripts. Uses of limited extent are specified as incidental.

Usage "on the line"

$a_1, a_2, \ldots, a_k$ Incidental constants (abstract only.)

A A choosable constant (sections 16, 33, 37)

b An incidental constant (section 9 only)

c An incidental constant (section 9 only), a choosable constant (section 16, 33, 37)

C, C_3, C_{21}, C_a, etc. Capacitance of element indicated by subscript (in delay line example). (section 10ff)

D Abbreviation for $\varphi'''-\varphi''\varphi''$. (section 16 only)

E Abbreviation for $\varphi''\varphi''$. (section 15 only)

$f(-,-,\ldots,-)$ A function of k arguments. (sections 41, 29, 15ff)

$f_a, f_{bc}, f_3, f_{112}$, etc. Values of the indicated derivatives of $f(-,-,\ldots,-)$ at the average point -- i.e., with each argument of $f(-,-,\ldots,-)$ at its average value. (sections 15ff)

$g(-,-,\ldots,-)$ A function of k arguments. (sections 41, 29, various)

$g_a, g_{bc}, g_3, g_{112}$, etc. Derivative values of $g(-,-,\ldots,-)$ at the average point (cp.f_a, etc. above)(sections 7ff)

$g^{(1)}(-,-,\ldots,-)$ A specific incidental function of k arguments. (section 7 only)

G, G_a Relative fifth moments about the average. (sections 2,3,5 and 21-23).

$h(-,-,\ldots,-)$ A function of k arguments. (section 41) 29, various)

$h_a, h_{bc}, h_3, h_{112}$, etc. Derivative values of $h(-,-,\ldots,-)$ at the average point (cp.$f_a, \ldots$, etc. above). (sections 3ff)

$H(-,-,-)$ Attenuation as function of W_A, W_B, W_C (section 27 only)

j Number of sections of delay line (sections 10, 11, 24ff).

k Number of arguments in response functions = number of individual variables treated as affecting response. (sections 41, various)

L, L_3, L_{2i-1}, L_a, etc. Inductance of element indicated by subscript (in delay line example. (sections 10, 11, 24ff.)

p See exponent in section 44.

p(z)dz Generic probability density of z. (section 2 only)

q_a Incidental deviation of w_a from its average. (section 23 only)

q_{ab}, etc. Incidental expressions (section 33 only)

R_A, R_B, R_C Values of resistance in attenuator example (sections 2, 27ff)

s_{ab}, s_{aab}, etc. Relative derivative values of f, e.g. $s_{ab} = f_{ab}/f_a f_b$. (sections 15ff, 33ff)

t_{ab}, f_{aab}, etc. Relative derivative values of g, e.g. $t_{ab} = g_{ab}/g_a g_b$. (section 8ff)

$u_1, u_2, \ldots, u_k$ A set of individual (= component) variables). (sections 41, 29, 15ff)

$u_a, u_b, \ldots$ General examples of $u_1, u_2, \ldots, u_k$. (various)

v A chance quantity (Part V only)

$v_1, v_2, \ldots, v_k$ A set of individual (= component) variables (sections 41, 29, 6ff)

$v_1^{(1)}, v_2^{(1)}, \ldots, v_k^{(1)}$ An incidental set of particular individual (= component) variables (section 7 only)

$v_a(u_a)$ A change in terms of the ath individual variable (sections 15, 29, 30)

$v_{10}, v_{20}, \ldots, v_k$ Incidental notation for the average values of $v_1, v_2, \ldots, v_k$. (section 9 only)

$v_{11}, v_{21}, \ldots, v_{k1}$ Incidental notation for the coordinates of the new average point (section 9 only).

v_{2i-1}, v_{2i} Measures of $\sqrt{L_{2i-1}}$ and $\sqrt{C_{2i}}$ respectively (sections 25, 26 with differing usage.)

$v_a^{i}, v_a^{ii}, v_a^{iii}, v_a^{iv}$, etc. Successive derivative values of $v_a(u_a)$ at the average point. (sections 30ff.)

v_A, v_B, v_C Measures of deviation of input shunt conductance, series resistance and output shunt conductance, respectively, from their nominal values (in the attenuator example) (sections 12, 28)

V_A, V_B, V_C Other similar measures (section 28 only)

$w_1, w_2, ---, w_k$ A set of k individual (= component) variables (sections 41, 29, various)

w_A, w_B, w_C Measures of deviation of element resistances from nominal values (in attenuator example)(sections 12, 27)

W_A, W_B, W_C Similar measures (section 27 only)

x A chance quantity (Part V only)

$x_1, x_2, \ldots, x_k$ A set of k individual (= component) variables. (sections 41, various.)

y A response variable (sections 41, various) A chance quantity (Part V only).

Y_A, Y_C Conductances of shunt elements (in attenuator example). (sections 12, 28).

z A response variable. (sections 41, various) A chance quantity, (Part V only)

α Nominal attenuation (in the attenuator example) (Sections 12, 27, 36, 37)

β_1, β_2 Dimensionless functions of higher moments. (Standard. Sections 4, 14)

β, γ, δ Specific rational functions of α (in the attenuator example) (sections 12, 27, 28, 36).

$\gamma, \gamma_a, \gamma_b$, etc. Relative third moments (in the attenuator example this meaning applies only to γ's with subscripts.) (Moderately used, section 2ff.)

γ_1, γ_2 Dimensionless functions of higher cumulants (standard. Sections 4, 13.)

Γ, Γ_a Relative fourth moments. (Transient, Section 2ff.)

Δ Deviation of t_{ab} from $1/\alpha$ (incidental). (section 26 only)

δ, δ^* "small" deviations from nominal specifying upper tolerances (sections 1, 3, 4)

$\varepsilon, \varepsilon^*$ "small" deviations from nominal specifying lower tolerances (section 1)

η_{2i-1}, η_{2i} Coefficients of variation of $\sqrt{L_{2i-1}}$ and $\sqrt{C_{2i}}$, respectively (in delay line example). (sections 11, 26)

κ_i ith cumulant = ith seminvariant. (standard, section 4 only)

μ_i ith moment about the average [about the mean]. (standard, sections 2, 4)

μ_i' ith moment about the origin (standard, section 2 only)

$v_1, v_2, \ldots, v_k$ Displacement of average point (in response units). (section 9 only)

$\varphi(-)$ Function expressing a transformation of response (section 15, 16, 29ff.)

$\varphi^{i}, \varphi^{ii}, \varphi^{iii}, \varphi^{iv}$, etc. Successive derivative values of $\varphi(-)$ at the new average point (sections 15, 16, 30ff.)

$\psi_a(w_a)$ Incidental change in terms of the ath individual variable. (section 7 only)

Σ^* Sign of restricted summation. (Discussed in section 38)

σ, σ_a, etc. Standard deviation = square root of variance = root-mean-square deviation. (standard, section 2ff)

Σ Sign of unrestricted summation. (standard, various)

τ_a RMS value of individual contribution to response = $f_a \sigma_a$ = $f_a (\mathrm{var}\ x_a)^{1/2}$. (Section 8ff.)

43. Index of notations used as subscripts.

a Generic subscript, usually ranging from 1 to k.

A Identifies input shunt element in attenuator example (sections 12, 27ff)

b Like a, but ordinarily distinct. (various)

B Identifies series element in attenuator example (sections 12, 27ff)

c Like a, but ordinarily distinct from both a and b. (various)

C Identifies output shunt element in the attenuator example. (sections 12, 27ff)

i Index ranging from 1 to j. (sections 10, 11, 24ff)

O Corresponding to average point. (various)

o Corresponding to average point. (abstract)

44. Index of notation used as superscripts.

i,j Incidental exponents. (section 21 only)

m,n Incidental exponents. (section 21 only)

p Exponent in power transformation. (sections 16, 33, 37)

p,q Incidental exponents. (section 21 only)